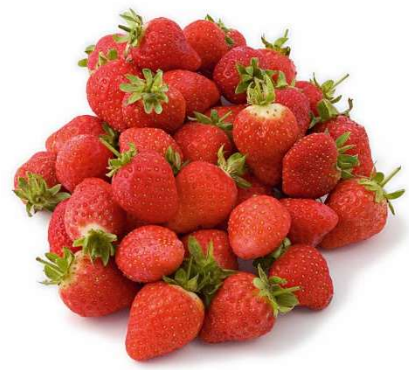
Chapel Hill Math Circle
Session 10 – April 25, 2026:
The Game of Nim
Beginners' Group (grades 1-3), 10:30-11:30a
Mr. Barman – dilip@trianglemathinstitute.com

Supplies needed per person: 25 objects like coins, dried beans, or cereal pieces

This is our last day of Math Circle for this school year, and I hope it has been fun. Let's end with a game called Nim.

The Game of Nim



There is an old game that is said to have come from ancient China where two players take turns removing objects from a pile[1]. The winner is the one who takes the last object, leaving nothing for the opponent. In 1901, about 120 years ago, a teacher from Harvard University, Charles Bouton, gave the name "Nim" to this game.

Let's play! You will need a partner and will have to decide who starts.

1. Put a pile of 25 items (like maybe the strawberries here) in between the players.
2. The first player can take 1, 2, 3, or 4 items from the pile.
3. The second person now takes 1, 2, 3, or 4 items from the pile.

Play continues flipping back and forth between players. The winner is the person who takes the last item - that is, the loser has nothing to take.

Can you Predict Who Will Win? Assuming you have a "worthy opponent" …

After you play for a while, can you come up with any strategies to make sure that you win? If you think that you have figured it out, try starting not with 25 but with 24 items.

Do you ever play a game with a younger brother or sister and maybe let him or her win? When we look at games from a mathematical perspective we assume that everybody wants to win and follows the rules to try their best to win. I call this having a "worthy opponent" – which I am not when I play board games with my family as I often don't take the best moves to "hurt" others.

If you think you have figured it out, try starting not with 25 but with 24 items. Assuming a "worthy opponent", who wins if you start with 25 items? 24 items? Can you predict the outcome with other starting number of items?

[1] picture by Alda Rizky Nur Afida from commons.wikimedia.org/wiki/File:Strawberry_pile.jpg, accessed Oct. 17, 2022, and sharable via a Creative Commons Attribution-Share Alike 4.0 International license

Hint: How to Derive a Strategy

Many of you probably had some ideas but may not have come up with a sure-fire strategy. When we're stuck it's worth solving easier problems. Let's see if that can help us here. Let's build a table of easy cases.

If you have this many objects …	Then …
0	You lost
1, 2, 3, or 4	You won as you would take them all and leave your opponent with none
5	You will lose; you must take 1, 2, 3, or 4 so would leave 4, 3, 2, or 1. In all cases your worthy opponent would in their next turn take all that remain.
6	You will win after noting that if you take 2, 3, or 4 your opponent will win by taking all that are left. So you take just 1, leaving your opponent with 5 on their turn – they are forced to leave 1, 2, 3, or 4 and you can take all on your turn.

Does this hint help you to figure out what to do? If you have 25 pieces should you start the game or let your opponent start? What if you have 24 pieces? 20? 10? How about 300 pieces?

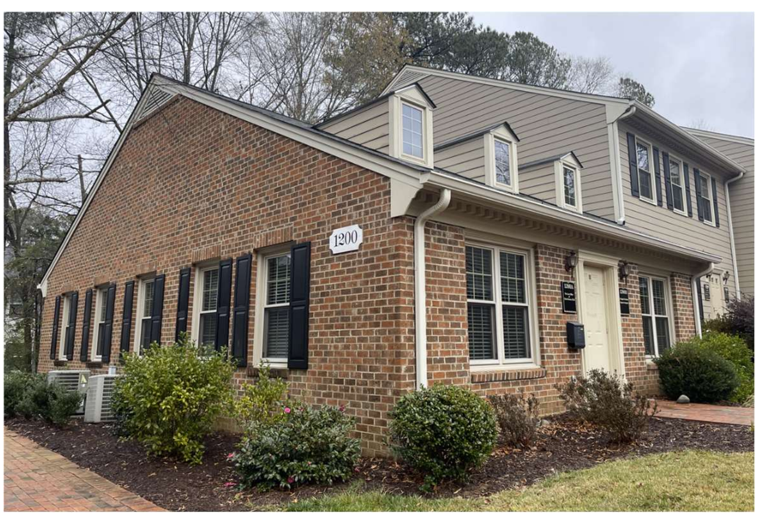
Try some variations

Please continue to play and think of this problem over the next week. Here are more ideas:

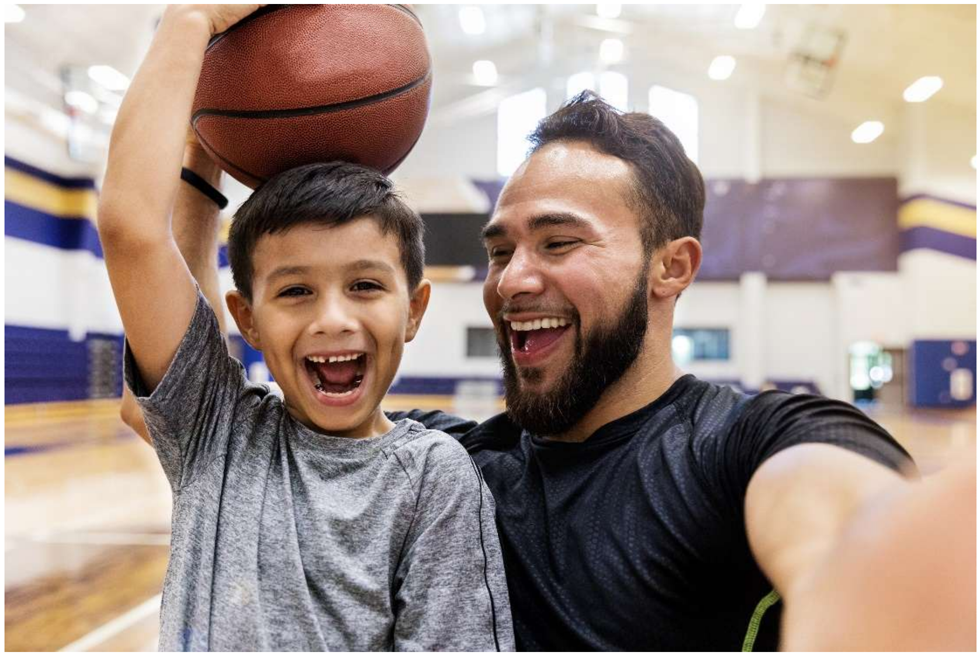
- Try a second version where on each turn a person can take 1, 2, or 4 items – but not 3. How does that change things?
- How about a third version where you always have a choice of passing and taking nothing?

Did you know that my school bought a new building? Last week we had some students bring hammers to begin the demolition before this summer's construction. It's going to be beautiful and I'll invite you to an open house in August. Have a great summer!

Enjoy
Mr. Barman



Notes for Parents



Math circle went quickly! I love working with your children. In the past I have organized a K-3 math circle and might do it again next summer (not this one) – especially since my school now has its own building.

Speaking of my school, Mathematics Institute of the Triangle, I expect in 2026-7 to teach level 2 (levels are more or less grades but one assesses into a particular level) likely Thursdays 3:30-5:30p, level 3 Wednesdays 4:15-6:15p, and level 4 Fridays 4:30-6:30p. We also will offer a programming class and one or two language arts classes. Our tentative schedule so far is below. Please contact me if you are interested in any of these classes to schedule an assessment. I'll send an invitation in August for an open house for the brand new location; it's going to be beautiful!

	M	Tu	W	Th	F	Sa	Su
3-4:20p					L1?		
3:30-5:30p				L2?			L5
4-6p	SAT/ACT prep						
4-6p							
4:15-6:15p			L3				
4:30-6:30p					L4		
6-8p			Competition Math	Algebra			Prealg
6-8p				Interm Alg			Geom

Nim is an interesting game and a great way for children to think about strategy and an early look into multiples. Let's see how the analysis can go. Here is the table that I shared with the students; I will highlight winning positions in green (apologies to any of you who are seeing this printed in black and white! (tell you what, I'll ~~cross out~~ losing positions)).

If you have this many objects …	Then …
~~0~~	You lost
1, 2, 3, or 4	You won as you would take them all and leave your opponent with none
~~5~~	You will lose; you must take 1, 2, 3, or 4 so would leave 4, 3, 2, or 1. In all cases your worthy opponent would in their next turn take all that remain.
6	You will win after noting that if you take 2, 3, or 4 your opponent will win by taking all that are left. So you take just 1, leaving your opponent with 5 on their turn – they are forced to leave 1, 2, 3, or 4 and you can take all on your turn.

Just as a reminder, there is a pile of items in the middle of two players and each in turn takes 1, 2, 3, or 4 items; they can't pass. The first person to have nothing left to take loses.

The hint shows that it's best in this problem to start with small cases. Let's continue the table.

7	Okay, now we see that a goal is to NOT leave our opponent with 4 or less items as they will win. Another observation is that analyzing the opponents' position is equivalent by symmetry to analyzing your own – i.e., whichever player has 0 loses, whichever has 1-4 wins, whichever has 5 loses, etc. as per the first part of the table. So we want to force our opponent to a losing position – i.e., to 5 (taking 1 puts them in the position of 6, a winner for them; taking 2 gives them the position 5 which is a loser for them; etc.). So 7 is a winner for us as we take 2 to force our opponent's hand. This is a good time to review the Worthy Opponent idea. The opponent or even one's self may not prove worthy especially while learning the rules of the game and understanding the implications of one's moves. Even a smart first grader (or adult!) might take 3 when faced with a pile of 4. Our table and predictions are based on players who know what they are doing and who want to win.
8	We have at least one move (taking 3 leaving them with 5 – the only winning move for you so here at least = exactly) that forces them to a losing position so you win.
9	9-1=8, 9-2=7, and 9-3=6 all put the opponent in a winning position but if we take 4 we put them in a losing position of 9-4=5, so that's what we do and we win.
~~10~~	Oops! No matter what we do we put the opponent in a winning position so we lose; the only outcomes of our turn are to end up having 9, 8, 7, or 6 for the opponent, all of which are winning positions for her.

Aha! There's a pattern. If we can get our opponent to have any multiple of 5, they will be stuck. To be careful to prove this to ourselves in an inductive style, we see that's clearly the case for 0 and 5. How can we get our opponent to 5 with larger numbers? If we have 5+1, 5+2, 5+3, or 5+4 – i.e., we win with 6, 7, 8, or 9. 10 is a loser. We repeat this argument assuming it's true for 5a where a is a positive integer and apply the same logic 5(a-1) times to reduce our problem to the case simply of 5a-5(a-1) = 5, which we've solved.

So we are guaranteed (assuming smart and motivated "worthy" players) to win if we start with any number that is not a multiple of 5. Playing with 25 (or any multiple of 5) tokens? Then be a gentleperson and insist that your opponent go first. Otherwise be chivalrous and offer to go first to show your opponent how the game is played!

This may be a difficult solution for your student to derive; don't be surprised or at all disappointed if they don't "solve" this. Hopefully they find the game fun and can keep playing occasionally; maybe then one day they'll have an "aha" moment.

I discourage you from sharing the "solution" and using terms like *multiple*. My best reasonable hope is that they see the inductive logic and realize that any number that they can get to by skip counting by 5 is a loser.

The variations that I suggested of starting with a different number of tokens are just ways for them to experiment to come up with the thought that they should avoid starting if there are a multiple of 5 tokens. What if they start with 300 pieces? It's not a big stretch to have smart 2nd and 3rd (and maybe 1st) graders see that 300 is a number that you would get to by skipping by 5; that was my "big number" variation.

The other variations are more interesting. What if you could take 1, 2, (NOT 3), or 4 tokens on each turn? We could build a table to illustrate why you lose with any multiple of 3 and otherwise win, but here is another way – just enumerate from 0 until you get the logic:

Lose				3	
Win	0	1	2		4

Does this make sense so far? Once you agree with it, let's continue from 5.

Lose		6 (forced to give opponent 5, 4, or 2)			9
Win	5 (e.g., take 2 to force opponent to losing position of 3)		7 (e.g., take 1)	8 (e.g., take 2)	

As to the last variation, I love having students consider the option of passing. To be blunt, this would be poor game design and teaches another lesson – games should be interesting. We have the option of deadlock with each person passing. The reason I asked students to consider this is to see if they realize the silliness of the option of passing and to see if they can take a germ of a good approach in it and suggest it as their better idea.

Maybe they come up, for example, with a rule that one can always pass but if one passes, the opponent cannot pass on the next turn. The analysis is a bit more complex; the first person who would lose would simply pass and force the other person to lose. With this rule you want to “play to lose” and do the opposite of what was deduced earlier.

Another improvement with passing might be that one could have a wildcard privilege of passing at most once but only if you started the game – or only if you were second – or only if you are older – or only if you roll a dice and … The possibilities are endless and with each of these any analysis your child can try would be interesting!

I hope to have other game sessions perhaps next year. One is the Lame Rook problem. A rook is in the top right of an 8x8 chessboard and can move any number of spaces down, to the left, or diagonally to the left and down - but never off an edge. The goal is to end up on the bottom left square from which the opponent can't move. What are winning and losing initial positions of the rook?

I love talking about where mathematical ideas come from. You might find the 36-page “The Prehistory of Nim Game” by Lisa Rougetet of interest; it’s downloadable at gathering4gardner.org/g4g11gift/Rougetet_Lisa-Prehistory_of_Nim.pdf (by the way, Gathering 4 Gardner itself is a fascinating conference that I learned about through math circle!).

Till next time! Happy summer!

Dilip

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