Chapel Hill Math Circle Exit Ticket:
April 25, 2026: The Calkin–Wilf Tree and Its Siblings

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- Was the this topic appropriately challenging relative to your background? That is, was the topic neither too elementary nor inaccessibly advanced?
- How could we improve this worksheet for future sessions?
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- Was there anything you thought was too difficult?
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- Are there any topics you would be interested in seeing us cover in the future?

The Calkin–Wilf Tree and Its Siblings

Abstract

Two themes are common in mathematics: simple rules can produce rich structure, and seemingly unrelated ideas are often unified. This week's session will illustrate both principles. We begin with the *Calkin–Wilf tree*, a *binary tree* of positive rational numbers with a simple defining rule that generates intricacy, as well as the related *Calkin–Wilf sequence*. One sibling of the Calkin–Wilf tree is the *Insert Sum Here* sequence-of-sequences. It likewise has a simple definition yielding many remarkable properties, and it is directly related to the Calkin–Wilf tree. The final sibling we introduce is *Stern's diatomic sequence*, an integer sequence with fractal properties, and it has direct connections to both the Calkin–Wilf tree and the Insert Sum Here sequence-of-sequences.

Background needed: Prerequisites include familiarity with basic algebra and inequalities. *Mathematical induction* and *proof by infinite descent* will be valuable proof techniques, especially for providing rigorous proofs. Several concepts from number theory—e.g., the *greatest common divisor (gcd)* of two integers, the *Euclidean algorithm* for computing gcd, the *binary* or *base-two* representations of integers, and *continued fractions*—will also all be useful. Properties of the *determinant* from linear algebra (as an *alternating multilinear* function) will be useful for several exercises in Section 2, especially for the $n \times n$ cases with $n \geq 3$.

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Contents

0 Warmup

As prerequisites for this session, it will help to answer the following first. You do *not* need to know these answers already, and many questions will be revisited later in the worksheet.

0.1. What is *mathematical induction*?

Note: Familiarity with mathematical induction will not be strictly necessary for this session, but it may be a useful method for providing rigorous proofs as solutions to some of the exercises below.

0.2. What is the *Euclidean algorithm*?

0.3. What is the *binary* or *base-two* notation of integers?

0.4. What is the *Fibonacci sequence*?

Note: The Fibonacci sequence was the advanced group's topic for sessions on February 11 and February 25, 2023.

0.5. Let n be a positive integer such that $\sqrt{n}$ is an irrational number. If a, b, c, d are integers, and if

$$x := \frac{a + b\sqrt{n}}{c + d\sqrt{n}}, \tag{0.1}$$

then what does it mean to *rationalize* x? What is the method for rationalizing such a number?

Remark. We naturally wish to exclude any cases of dividing by zero for the expression in (0.1). Note that since c, d are integers and $\sqrt{n}$ is *irrational*, the denominator $c + d\sqrt{n} = 0$ if and only if $c = d = 0$.

1 The Calkin–Wilf Tree

1.1 Discussion

This section introduces our first object of study, the Calkin–Wilf tree.

Definition 1.1.1 (The Calkin–Wilf Tree)**.** *The Calkin–Wilf tree*[1] is a graph-theoretic *tree*—i.e., a *connected* graph[2] with no *cycles*—with positive rationals of the form $\frac{i}{j}$ at each node, where i, j are positive integers, and the fraction is in lowest terms.[3] The tree is generated by $1 = 1/1$ at its apex, along with the following rule: for each entry $\frac{i}{j}$ in the tree, there are two *children* given by the formulas

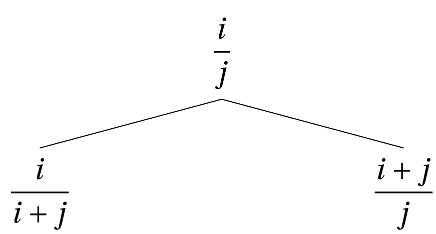

Figure 1.1.1: Generating the Calkin–Wilf tree.

That is, for any entry $\frac{i}{j}$ in the Calkin–Wilf tree, its left and right *children* are

$$\text{left}\left(\frac{i}{j}\right) := \frac{i}{i+j} \tag{1.1.1}$$

$$\text{right}\left(\frac{i}{j}\right) := \frac{i+j}{j}, \tag{1.1.2}$$

[1] Named for American mathematicians *Neil Calkin* (b. 1961) and *Herbert S. Wilf* (1931–2012). The pair introduced the tree in [4], a four-page paper that should be very accessible to those in the advanced group, especially after this session!

[2] This is a graph in the sense of *graph theory*, the setting for multiple past advanced group sessions, most recently on October 25 and November 8, 2025.

[3] That is, $\gcd(i, j) = 1$. Note that $\text{left}\left(\frac{i}{j}\right)$ and $\text{right}\left(\frac{i}{j}\right)$ are defined, *a priori*, only when $i, j > 0$ and $\gcd(i, j) = 1$. From the defining formulas in (1.1.1)–(1.1.2), it is clear that these formulas remain valid even when i, j share a nontrivial common factor.

respectively.

Each child will itself have its own left and right children, too, and the tree expands downward with infinitely many rows.

Notation. We shall also use the notation

$$q \overset{\mathrm{L}}{\mapsto} q_{\text{left}} \quad \text{if and only if} \quad \operatorname{left}(q) = q_{\text{left}}, \text{ and} \tag{1.1.3}$$

$$q \overset{\mathrm{R}}{\mapsto} q_{\text{right}} \text{ if and only if } \operatorname{left}(q) = q_{\text{right}}, \tag{1.1.4}$$

since this may be more useful to trace the branches of the family tree an element q in the tree.

Via this rule, the first three rows of the Calkin–Wilf tree are given in Figure 1.1.2:

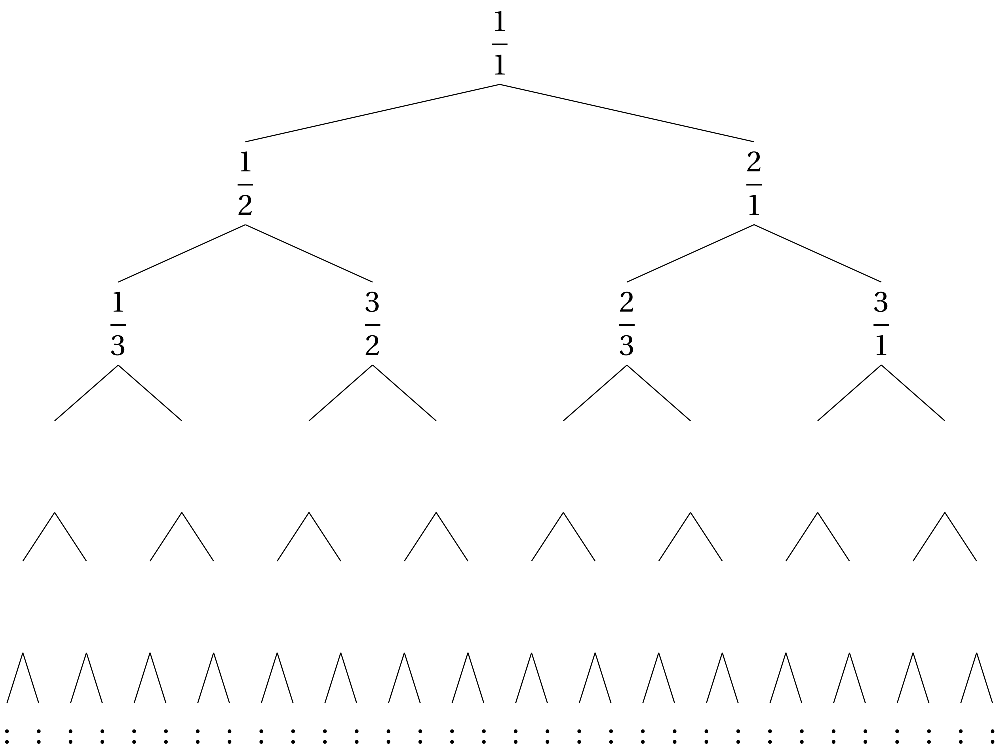

Figure 1.1.2: The first three rows of the Calkin–Wilf tree. (See also Exercise #1.2.1, which asks you to determine the entries in the empty nodes.)

It can be useful to consider the elements of the the Calkin–Wilf tree as a sequence, ignoring its graph-theoretic structure as a tree:

Definition 1.1.2 (The Calkin–Wilf Sequence)**.** The *Calkin–Wilf sequence* is the sequence (c_n) formed by taking the elements from the Calkin–Wilf tree in order, starting with $\frac{1}{1}$ in the top row, then proceeding left-to-right in the next row, and continuing indefinitely.

Here, c_n denotes the nth term in the sequence (c_n). From Figure 1.1.2, the first several terms of the Calkin–Wilf sequence, in order, are

$$(c_n) = \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \ldots, \tag{1.1.5}$$

so $c_1 = \frac{1}{1}$, $c_2 = \frac{1}{2}$, $c_3 = \frac{2}{1}$, and so on.

Definition 1.1.3. Let q be an element in the Calkin–Wilf tree.

(a) If q lies in the Calkin–Wilf tree, then q' is a *child* of q if either $q' = \text{left}(q)$ or $q' = \text{right}(q)$.

(b) If q lies in the Calkin–Wilf tree, then a[4] *parent* of q, denoted $\text{parent}(q)$, is a rational number such that either

$$\text{left}\big(\text{parent}(q)\big) = q, \text{ or} \tag{1.1.6}$$

$$\text{right}\big(\text{parent}(q)\big) = q. \tag{1.1.7}$$

(c) Let q, q' be *distinct* entries of the Calkin–Wilf tree. Then q, q' are *siblings* if and only if they have the same parent, meaning $\text{parent}(q) = \text{parent}(q')$.

(d) The rational number q_{d} is a *descendant* of q if and only if q_{d} arises from a sequence of successive children of q.

That is, $\text{parent}(q)$ is a rational number such that q is one of the two children of $\text{parent}(q)$.

(e) The rational number q_{a} is an *ancestor* of q if and only if q is a descendant of q_{a}.

Example 1.1.4. Since

$$\text{right}\left(\frac{1}{2}\right) = \frac{3}{2}, \text{ therefore parent}\left(\frac{3}{2}\right) = \frac{1}{2}. \tag{1.1.8}$$

Example 1.1.5. Since

$$\frac{2}{3} = \text{left}\left(\text{right}\left(\frac{1}{1}\right)\right), \tag{1.1.9}$$

this means that $\frac{2}{3}$ is a descendant of $\frac{1}{1}$. Conversely, $\frac{1}{1}$ is an ancestor of $\frac{2}{3}$.

[4] In Exercise #1.2.12, we shall show that for a positive rational number q in the Calkin–Wilf tree, $\text{parent}(q)$ is uniquely determined by q. *A priori*, though, we must say *a* parent of q rather than *the* parent of q.

1.2 Exercises

Note: Exercises #1.2.10–1.2.12 below are taken from Section 1 of [4].

1.2.1. Complete rows 4–5 in Figure 1.1.2 for the Calkin–Wilf tree. If we begin by numbering top row as row #1, how many terms in general appear in the mth row of the Calkin–Wilf tree? How many in all the first m rows?

Challenging: Can you form a conjecture for the *sum* of the elements in the mth row of the Calkin–Wilf tree?

Remark. The video in [11] shows the Admissions and Outreach Coordinator in maths at the University of Oxford working through the challenge question.

1.2.2. For each of the following positive rational numbers q, compute their left and right children in the Calkin–Wilf tree.

(a) $q := \frac{5}{6}$

(b) $q := \frac{15}{7}$.

(c) $q := \frac{31}{101}$

(d) $q := \frac{1}{n}$, where n is a positive integer

1.2.3. For each of the following positive rational numbers q, compute their parents in the Calkin–Wilf tree.

(a) $q := \frac{5}{6}$

(b) $q := \frac{15}{7}$

(c) $q := \frac{31}{101}$

(d) $q := \frac{1}{n}$, where n is a positive integer

1.2.4. **Open question:** What other structure do you notice about the Calkin–Wilf tree? What conjectures can you formulate? Can you prove them?

1.2.5. Prove that for all positive rational numbers q,

$$0 < \operatorname{left}(q) < 1, \text{ and} \tag{1.2.1}$$

$$\operatorname{right}(q) > 1. \tag{1.2.2}$$

1.2.6. Let $\frac{i}{j}$ be the lowest-terms fraction representation of some positive rational number q. Further, assume that q lies in the Calkin–Wilf tree.

(a) Prove that

$$\operatorname{left}(q) = \frac{q}{q+1} = \frac{1}{1+\frac{1}{q}}. \tag{1.2.3}$$

(b) Prove that

$$\operatorname{right}(q) = q + 1. \tag{1.2.4}$$

1.2.7. To verify that a particular positive rational number q lies in the Calkin–Wilf tree, that is equivalent to show that either $q = 1 = \frac{1}{1}$, or that 1 is an ancestor of q.

Example 1.2.1. Consider the example $q := \frac{12}{5}$. We shall show that q lies in the CWT by showing that $\frac{1}{1}$ is an ancestor for q.

$$\begin{aligned}
\frac{12}{5} &= \operatorname{right}\left(\frac{7}{5}\right)\\
\frac{7}{5} &= \operatorname{right}\left(\frac{2}{5}\right)\\
\frac{2}{5} &= \operatorname{left}\left(\frac{2}{3}\right)\\
\frac{2}{3} &= \operatorname{left}\left(\frac{2}{1}\right)\\
\frac{2}{1} &= \operatorname{right}\left(\frac{1}{1}\right), \text{so}\\
\frac{12}{5} &= \operatorname{right}\left(\operatorname{right}\left(\operatorname{left}\left(\operatorname{left}\left(\operatorname{right}\left(\frac{1}{1}\right)\right)\right)\right)\right).
\end{aligned}$$

Equivalently,

$$\frac{1}{1} \overset{\text{R}}{\mapsto} \frac{2}{1} \overset{\text{L}}{\mapsto} \frac{2}{3} \overset{\text{L}}{\mapsto} \frac{2}{5} \overset{\text{R}}{\mapsto} \frac{7}{5} \overset{\text{R}}{\mapsto} \frac{12}{5}, \tag{1.2.5}$$

meaning $\frac{12}{5}$ is a descendant of $\frac{1}{1}$, as claimed.

Without expanding the tree, prove the following positive rationals lie in the Calkin–Wilf tree.

Hint: Modify the calculations of Example 1.2.1 for the following values of q to show that each q is a descendant of $\frac{1}{1}$.

(a) $q := \dfrac{5}{6}$

(b) $q := \dfrac{15}{7}$

(c) $q := \dfrac{31}{101}$

(d) $q := \dfrac{1}{n}$, where n is a positive integer

1.2.8. Formulas (1.1.1)–(1.1.2) give the rules to form the right and left children of an entry $\frac{i}{j}$ in the Calkin–Wilf tree. Assuming $\frac{i}{j}$ is a positive rational number in the Calkin–Wilf tree, show that $\frac{i}{j}$ is a left child if and only if $\frac{i}{j} < 1$, and $\frac{i}{j}$ is a right child if and only if $\frac{i}{j} > 1$.

Conclude that

$$\operatorname{parent}\left(\frac{i}{j}\right) = \begin{cases} \dfrac{i}{j-i}, & \text{if } \dfrac{i}{j} < 1, \\ \dfrac{i-j}{j}, & \text{if } \dfrac{i}{j} > 1, \text{ and} \\ \text{does not exist}, & \text{if } \dfrac{i}{j} = \dfrac{1}{1}. \end{cases} \tag{1.2.6}$$

1.2.9. (a) Let $q > 0$ be a rational number. Prove that q appears in row m if and only if $\frac{1}{q}$ appears in row m. That is, the mth row is closed under taking reciprocals.

Hint: Consider inducting on the row number m, and compare $\text{parent}(q)$ to $\text{parent}\left(\frac{1}{q}\right)$.

(b) Let q be a rational number with $0 < q < 1$. Then q appears in row m if and only if $1 - q$ appears in row m. That is, for a rational number strictly between 0 and 1, its "complement" in this sense also appears in the same row.

Hint: Consider inducting on the row number m, and for $0 < \frac{i}{j} < 1$, compare $\text{parent}\left(\frac{i}{j}\right)$ to $\text{parent}\left(1 - \frac{i}{j}\right) = \text{parent}\left(\frac{j-i}{j}\right)$. The result from Exercise #1.2.9(a) will likely be helpful here, too.

1.2.10. Let $\frac{i}{j}$ be a positive rational number appearing in the Calkin–Wilf tree generated by $\frac{1}{1}$ and successive applications of formulas (1.1.1) and (1.1.2) in Definition 1.1.1. Prove that $\gcd(i, j) = 1$, meaning $\frac{i}{j}$ is in lowest terms.

1.2.11. Prove that every positive rational number appears *at least once* in the Calkin–Wilf tree.

Hint: Consider a proof-by-contradiction. Take the set of positive rationals q that do *not* appear in the tree. If q is such a "bad" fraction, what can we conclude about $\text{parent}(q)$?

1.2.12. Prove that every positive, rational q appears *at most once* in the Calkin–Wilf tree. Equivalently, prove that if q lies in the Calkin–Wilf tree, then q is not repeated.

Combining this result with that of Exercise #1.2.11, show how this provides an explicit enumeration of the positive rational numbers. In other words, the set of positive rationals is *countably infinite*, meaning it is in *one-to-one correspondence* (or *bijection*) with the set of all positive integers.

Hint: As in Exercise #1.2.11, consider a proof-by-contradiction. Take the set of positive rationals q that appear *more than once* in the tree. If q is such a "bad" fraction, what can we conclude about parent(q)?

Remark. The advanced group explored countability and bijections in "Set Theory and Cardinality" from November 22, 2025.

1.2.13. Let (c_n) be the Calkin–Wilf sequence in Definition 1.1.2. Prove that for any two consecutive terms c_n and c_{n+1} expressed in lowest terms, the denominator of c_n is the numerator of c_{n+1}.

Hint: If c_n and c_{n+1} are the left and right children of the same rational q, then this follows immediately by formulas (1.1.1)–(1.1.2). What other cases are possible other than c_n and c_{n+1} being siblings to the same parent?

Remark. This result is taken from [14].

1.2.14. Using properties of the Calkin–Wilf tree, complete the following steps to establish a proof-by-contradiction that $\sqrt{2}$ is irrational. Clearly $\sqrt{2} > 0$. *Assuming* that $q := \sqrt{2}$ is a rational number, it follows by Exercises #1.2.11–1.2.12 that it must appear precisely once in the Calkin–Wilf tree.

Remark. This proof-by-contradiction strategy to prove irrationality can be used to prove that more general quadratic terms of the form $r + s\sqrt{d} > 0$, where r, s are rational, d is a positive integer, and $\sqrt{d}$ is irrational, are themselves irrational. For examples and generalizations in exploring quadratic irrationals, see [9], [7], and [12]

(a) Use Exercise #1.2.6 to show that under this hypothesis,

$$\sqrt{2} \overset{\text{R}}{\mapsto} \sqrt{2}+1 \overset{\text{L}}{\mapsto} \frac{\sqrt{2}}{2} \overset{\text{L}}{\mapsto} \sqrt{2}-1 \overset{\text{R}}{\mapsto} \sqrt{2}. \tag{1.2.7}$$

Note: It will simplify computations here by knowing how to rationalize expressions of the form $\frac{a+b\sqrt{2}}{c+d\sqrt{2}}$, where a, b, c, d are integers, as in Exercise #0.5 from the warmup.

(b) Show that Exercise #1.2.14(a) contradicts Exercise #1.2.12. Conclude that our original assumption is false, so $\sqrt{2}$ must be irrational.

2 "Insert Sum Here"

2.1 Discussion

Several years ago, Joshua Zucker directed the session "Insert Sum Here"[5]for *The Julia Robinson Mathematics Festival*. That session introduced involving a sequence-of-sequences which has a surprising amount of structure. This section describes the Insert Sum Here (henceforth "ISH") array, explores some of its properties, and makes connections to the Calkin–Wilf tree and sequence of Section 1.

The ISH array is a sequence of rows, each of which is a sequence of integers. The first row, R_1, is 1 1. The ISH array is then defined recursively: to form the $(m+1)$st row R_{m+1}, we take the previous row R_m. Between each pair of entries that were adjacent in R_m, in R_{m+1}, we insert their sum. For example, here are the first five rows of ISH:

[5] Zucker's original worksheet appears to be unavailable—or at least sufficiently difficult to find—online. Some background on the session is provided in a Numberplay entry of *The New York Times*' Wordplay blog [2].

$$\begin{array}{cccccccccccccccccc}
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 2 & 3 & 1 \\
1 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 1 \\
1 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 2 & 7 & 5 & 8 & 3 & 7 & 4 & 5 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array} \tag{2.1.1}$$

Formally, we have the following:

Definition 2.1.1 ("Insert Sum Here" Array)**.** The *"Insert Sum Here" array* is a set of terms $a(m,n)$ satisfying the following initial conditions and recursive formulas:

$$a(1,1) := 1 \tag{2.1.2}$$

$$a(1,2) := 1 \tag{2.1.3}$$

$$a(m+1,2n-1) := a(m,n) \tag{2.1.4}$$

$$a(m+1,2n) := a(m,n) + a(m,n+1). \tag{2.1.5}$$

For ordered pairs (m,n) such that $a(m,n)$ cannot be computed from (2.1.2)–(2.1.5), we say $a(m,n)$ is undefined.

Comparing Definition 2.1.1 to Equation (2.1.1), the general form of the ISH array in terms of $a(m,n)$ is

$$\begin{array}{cccccccccc}
a(1,1) & a(1,2) \\
a(2,1) & a(2,2) & a(2,3) \\
a(3,1) & a(3,2) & a(3,3) & a(3,4) & a(3,5) \\
a(4,1) & a(4,2) & a(4,3) & a(4,4) & a(4,5) & a(4,6) & a(4,7) & a(4,8) & a(4,9) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array} \tag{2.1.6}$$

Example 2.1.2. Let us use Equations (2.1.2)–(2.1.5) to compute R_2 and R_3. The first row is

$$R_1 = a(1,1) \quad a(1,2) \tag{2.1.7}$$

By our initial conditions, $a(1,1) = a(1,2) = 1$, so

$$R_1 = 1 \quad 1 \tag{2.1.8}$$

By definition,

$$R_2 = a(2,1) \quad a(2,2) \quad a(2,3) \tag{2.1.9}$$

By (2.1.4), the entries from R_1 become the odd-indexed entries in row 2:

$$R_2 = a(1,1) \quad a(2,2) \quad a(1,2) \tag{2.1.10}$$

By Equation (2.1.7), we already know the $a(1,n)$ entries from R_1:

$$R_2 = 1 \quad a(2,2) \quad 1 \tag{2.1.11}$$

Finally, by (2.1.5), $a(2,2) = a(1,1) + a(1,2)$, so

$$R_2 = 1 \quad 2 \quad 1 \tag{2.1.12}$$

Can you see how the rows R_3, R_4, and R_5 are formed in (2.1.1)?

Our goal in this section is to explore the structure of the ISH array and related mathematical structures and making connections to the Calkin–Wilf tree and sequence.

2.2 Exercises

2.2.1. Let R_m denote the mth row of the ISH array. In Equation (2.1.1), we are shown R_1 through R_5. Produce R_6. Can you also compute R_7?

2.2.2. Let m be a positive integer. How many elements are in R_m? That is, what is $|R_m|$? Consider whether you can produce a recursive formula, defined in terms values of $|R_j|$ for indices $j < m$. Using that recursive formula, or otherwise, can you produce a closed-form expression for $|R_m|$ depending on m alone?

Note: We are asking for how many entries lie in R_m, not how many *distinct* entries there are. For example, $|R_3| = 5$, and $|R_4| = 9$, even R_3 includes only 3 *distinct* values and r_4 has only 5 distinct values.

2.2.3. Let S_m denote the sum of all the entries in R_m. Compute S_m. As in Exercise #2.2.2, consider both recursive and closed-form solutions.

2.2.4. Prove that every row R_m is a *palindrome*.

2.2.5. Let m be any positive integer, and let n be a positive integer such that both $a(m,n)$ and $a(m,n+1)$ appear in R_m. Prove that $\gcd(a(m,n), a(m,n+1)) = 1$.

That is, prove that any two consecutive terms in a given row are *relatively prime* (or *coprime*).

2.2.6. Describe a relationship between the Calkin–Wilf tree and the ISH array. How does your answer compare to Exercise #3.2.4?

2.2.7. For each positive integer m, what is the largest element of R_m?

2.2.8. For each positive integer m, how many times does m appear in R_m? Does this number change if we consider how many times m appears in R_j, where j is any positive integer?

Potentially Challenging: How many times does 2026 appear in R_{2026}?

2.2.9. Let i, j be positive integers such that $\gcd(i, j) = 1$. Prove that i, j appear in that order as consecutive entries in some row R_m. Further, prove that this ordered pair appears precisely once as consecutive entries in some row.

Remark. Note that $\gcd(i, j) = \gcd(j, i)$. If $\gcd(i, j) = 1$, it follows that the ordered pairs (i, j) and (j, i) both appear as consecutive entries in some row (though the pairs do not, *a priori*, appear in the *same* row as each other).

Hint: Consider the rational number $\frac{i}{j}$ in the context of the Calkin–Wilf tree. Using the tree's properties, what can you conclude?

2.2.10. Consider the following 2×2 submatrices of the ISH array:

$$\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \tag{2.2.1}$$

$$\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} \tag{2.2.2}$$

$$\begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix} \tag{2.2.3}$$

We define the *determinant* of a 2×2 matrix by the formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} := ad - bc. \tag{2.2.4}$$

What do you notice about these 2×2 determinants? Can you prove whether this always holds for any determinant formed from a 2×2 submatrix of the ISH array?

2.2.11. **Challenging:** Extend Exercise #2.2.10 by considering 3×3, 4×4, and in general, $n \times n$ determinants. For example, the following is an example of a 3×3 submatrix of the ISH array:

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 2 \\ 4 & 7 & 3 \end{bmatrix} \tag{2.2.5}$$

Form a conjecture about the value of the determinant for any such $n \times n$ submatrix with $n \geq 3$. Can you prove it?

Note: As a starting point, how would you define determinants for $n \times n$ matrices, where $n \geq 3$ is a positive integer?

3 Stern's Diatomic Sequence

3.1 Discussion

Let us introduce a new sequence, defined recursively:

Definition 3.1.1 (Stern's Diatomic Sequence)**.** *Stern's diatomic sequence*,[6] denoted (s_n) for $n \geq 0$, is the sequence satisfying

$$s_0 := 0, \tag{3.1.1}$$
$$s_1 := 1, \tag{3.1.2}$$
$$s_{2n} := s_n, \qquad \text{and} \tag{3.1.3}$$
$$s_{2n+1} := s_n + s_{n+1} \tag{3.1.4}$$

for every nonnegative integer n.

Example 3.1.2. Let us compute s_n for $n = 0, 1, 2, \ldots, 5$.

- By (3.1.1), $\boxed{s_0 = 0}$.
- By (3.1.2), $\boxed{s_1 = 1}$.
- Note that $2 = 2 \cdot 1$. Combining (3.1.2) and (3.1.3), we have $s_2 = s_1$, so $\boxed{s_2 = 1}$.

[6]Named for *Moritz Stern (1807–1894)*, a German mathematician. The is also described as the *fusc* function, where $\text{fusc}(n) := s_n$, so named by Dutch computer scientist and mathematician *Edsger W. Dijkstra (1930–2002)* for the functions properties of obfuscation.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\cdots$
s_n	0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1	5	4	7	

Table 3.1.1: Values of s_n for small indices n.

- Note that $3 = 2 \cdot 1 + 1$. By (3.1.4), this means $s_3 = s_1 + s_2$. Combining this with the above, $\boxed{s_3 = 2}$.
- Since $4 = 2 \cdot 4$, since $s_2 = 1$, by (3.1.3) we also have $\boxed{s_4 = 1}$.
- Since $5 = 2 \cdot 2 + 1$, by (3.1.4), we have $s_5 = s_2 + s_3 = 1 + 2$. Therefore, $\boxed{s_5 = 3}$.

Continuing, Table 3.1.1 gives values for s_n for small n:

Remark. Note by (3.1.3), the even-indexed subsequence (s_{2n}) of (s_n) is the same as (s_n) itself! This self-similarity means the sequence (s_n) has fractal-like properties.

In this section, we consider the number of *hyperbinary* representations of a positive integer n, then show the connections between this number and the ISH sequence.

Definition 3.1.3. Let n be a nonnegative integer. A *hyperbinary* expansion of n is a sum of powers of two, used at most twice, that sum to n. For a given nonnegative integer n, we let $h(n)$ denote the number of distinct hyperbinary representations of n. By convention, $h(0) := 1$.

Example 3.1.4. Consider the following examples:

- $h(2) = 2$: we have the two hyperbinary decompositions $2 = 2^1 = 2^0 + 2^0$, and no other hyperbinary decompositions of 2 are possible.
- $h(6) = 3$, since $6 = 4 + 2 = 4 + 1 + 1 = 2 + 2 + 1 + 1$, and no other hyperbinary decompositions of 6 are possible.
- $h(7) = 1$, since $7 = 4 + 2 + 1$ is the unique hyperbinary representation of 7.
- $h(10) = 5$, since $10 = 8 + 2 = 8 + 1 + 1 = 4 + 4 + 2 = 4 + 4 + 1 + 1 = 4 + 2 + 2 + 1 + 1$, with no other hyperbinary decompositions of 10 possible.

3.2 Exercises

3.2.1. Complete Table 3.2.1 to compute $h(n)$ for these small indices n.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\cdots$
$h(n)$	1	1	2	1	3		3	1			5							]			$\cdots$

Table 3.2.1: Values of $h(n)$ for small indices n.

3.2.2. Let n be a nonnegative integer. Prove

$$h(2n+1) = h(n) \tag{3.2.1}$$

$$h(2n+2) = h(n) + h(n+1). \tag{3.2.2}$$

Conclude that for all $n \geq 1$,

$$h(n) = s_{n+1}. \tag{3.2.3}$$

3.2.3. Prove that for all positive integers n, $s_{2^n} = 1$ and $h(2^n - 1) = 1$.

3.2.4. What is the relationship between Stern's diatomic sequence, the hyperbinary counting function h, and the ISH sequences of the previous sections?

Hint: Compare Equations (3.2.1) and (3.2.2) to Equations (2.1.4) and (2.1.5). What can you conclude?

4 Continued Fractions and the Calkin–Wilf Tree

4.1 Discussion

Much of the structure of the Calkin–Wilf tree is revealed in terms of *continued fractions*, a method for expressing rational numbers as nested fractions.

Definition 4.1.1. Let q be a rational number. Then a *continued fraction expansion*[7] for q is an expression of the form

$$[a_0; a_1, \dots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}, \tag{4.1.1}$$

where a_0 is an integer, $a_1, \dots, a_n$ are *positive* integers, and n is a nonnegative integer.

Remark. By convention, we prefer not to take $a_n = 1$ unless $n = 0$ and $a_0 = 1$. Otherwise $[a_0; a_1, \dots, a_{n-1}, 1] = [a_0; a_1, \dots, a_{n-1}+1]$. Continued fractions of the form $[a_0; a_1, \dots, a_{n-1}, 1]$ are perfectly legitimate as expressions for continued fractions in general, but this alternate expression is considered nonstandard as "the" continued fraction representation for a rational number. For example, $[3] = 3$, and $[2;1] = 2 + \frac{1}{1} = 2 + 1 = 3$, but we typically think of $[3]$ as being "the" continued fraction expansion of 3.

Example 4.1.2. Let us express the continued fraction

$$[4; 1, 2, 3] := 4 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3}}} \tag{4.1.2}$$

as a rational number of the form P/Q, where P, Q are integers, Q is positive, and $\gcd(P, Q) = 1$. In other words, we want to express the continued fraction $[4; 1, 2, 3]$ as a fraction in lowest terms.

[7]Since the numerators are always 1 in (4.1.1), these are more specifically *simple* continued fractions. Continued fractions were our topic for the advanced group, most recently on January 18 and February 1, 2025.

Working from the bottom up, we have

$$4+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3}}}=4+\cfrac{1}{1+\cfrac{1}{\frac{7}{3}}}=4+\cfrac{1}{1+\frac{3}{7}}=4+\cfrac{1}{\frac{10}{7}}=4+\frac{7}{10}$$
$$\Rightarrow 4+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3}}}=\boxed{\frac{47}{10}},$$

so we may take $\underline{P := 47,\ Q := 10}$.

The following, an algorithm for computing the *greatest common divisor ("gcd")* of two integers, is highly relevant in computing continued fractions:

Theorem 4.1.3 (The Euclidean Algorithm)**.** *Let a, b be integers with $b \neq 0$. Let q_j, r_j be the unique quotients and remainders under the Division Algorithm such that*

$$a = bq_1 + r_1, \qquad \text{with } 0 < r_1 < |b| \tag{4.1.3}$$
$$b = r_1q_2 + r_2, \qquad \text{with } 0 < r_2 < r_1 \tag{4.1.4}$$
$$r_1 = r_2q_3 + r_3, \qquad \text{with } 0 < r_3 < r_2 \tag{4.1.5}$$
$$\vdots \qquad \vdots$$
$$r_j = r_{j+1}q_{j+2} + r_{j+2}, \qquad \text{with } 0 < r_{j+2} < r_{j+1} \tag{4.1.6}$$
$$\vdots \qquad \vdots$$
$$r_{n-2} = r_{n-1}q_n + r_n, \qquad \text{with } 0 < r_n < r_{n-1} \tag{4.1.7}$$
$$r_{n-1} = r_nq_{n+1}; \tag{4.1.8}$$

that is, n and r_n are defined so that r_n last nonzero remainder from successive applications of the Division Algorithm. Then

$$\gcd(a, b) = r_n. \tag{4.1.9}$$

That is, we divide a by b, then take the integer quotient q_1 and remainder r_1. Next, we divide b by r_1, taking integer quotient q_2 and remainder r_2. If we continue dividing each remainder r_j by the next remainder r_{j+1} to determine the next remainder r_{j+2}, then the final nonzero remainder r_n is equal to $\gcd(a, b)$, the greatest common divisor of a and b. The Euclidean algorithm is explored in more detail in our session of October 7, 2023.

Example 4.1.4. Consider the case $a := 47$, $b := 10$, and use the Euclidean algorithm to compute $\gcd(47, 10)$. Then we have

$$\begin{aligned} 47 &= 10 \cdot \underline{4} + 7 \\ 10 &= 7 \cdot \underline{1} + 3 \\ 7 &= 3 \cdot \underline{2} + \boxed{1} \\ 3 &= 1 \cdot \underline{3} + 0. \end{aligned}$$

and $\gcd(a,b) = \gcd(47,10) = \underline{1}$, the last nonzero remainder above.

Remark. Compare the underlined quotients in Example 4.1.4 to our solution to Example 4.1.2.

Corollary 4.1.4(a). *Let a, b be integers with $b > 0$. Then if q_i and r_i are integers as in* (4.1.3)–(4.1.8)*, we have*

$$q_1 = \left\lfloor \frac{a}{b} \right\rfloor, q_2 = \left\lfloor \frac{b}{r_1} \right\rfloor, \dots, q_n = \left\lfloor \frac{r_{n-2}}{r_{n-1}} \right\rfloor, \text{ and } q_{n+1} = \left\lfloor \frac{r_{n-1}}{r_n} \right\rfloor = \frac{r_{n-1}}{r_n},$$

and in general,

$$q_j = \left\lfloor \frac{r_{j-2}}{r_{j-1}} \right\rfloor \tag{4.1.10}$$

where $\lfloor x \rfloor$ denotes the greatest integer *or* floor function *of the real number x.*

Example 4.1.5. Express the rational number $-\frac{49}{23}$ as a continued fraction.

The natural starting point[8] is to compute a_0, the first coefficient in the continued fraction $[a_0; a_1, \dots, a_n]$ for $\frac{49}{23}$. Note that $a_1, a_2, \dots, a_n$ are all *positive* integers, so the expression $[a_1; a_2, a_3, \dots, a_n]$ will be positive and satisfy the inequality $[a_1; a_2, a_3, \dots, a_n] < 1$. Combining this, we conclude that

$$a_0 = \left\lfloor -\frac{49}{23} \right\rfloor, \tag{4.1.11}$$

whence $a_0 = -3$.

Beginning with a_0, we have

$$-\frac{49}{23} = -3 + \frac{20}{23} = -3 + \cfrac{1}{\cfrac{23}{20}} = -3 + \cfrac{1}{1 + \cfrac{3}{20}} = -3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{20}{3}}}$$

$$= -3 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{2}{3}}} = -3 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{\cfrac{3}{2}}}} = -3 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{2}}}},$$

and therefore

$$-\frac{49}{23} = [-3; 1, 6, 1, 2]. \tag{4.1.12}$$

[8] After all, we don't *a priori* know how many terms are in the continued fraction $[a_0; a_1, \dots, a_n]$ for $-\frac{49}{23}$, so it seems premature to start with the "bottom" entry a_n until we know how deep the bottom is.

4.2 Exercises

4.2.1. Compute the (standard) continued fraction expansions for the first 3–5 rows of the Calkin–Wilf tree.

Note: This exercise becomes a bit easier if you can first solve Exercise #4.2.2, especially if trying to compute the continued fractions for rows deeper into the Calkin–Wilf tree.

4.2.2. Let $q := \frac{i}{j}$ be a positive, rational number reduced to lowest terms. Further, let the continued fraction expansion for q be $[a_0; a_1, \ldots, a_n]$. What are the respective continued fraction expansions for the left and right children of q?

4.2.3. Let $q > 0$ be a rational number with continued fraction expansion $q = [a_0; a_1, \ldots, a_n]$. Define the *order* of q to be

$$\operatorname{ord}(q) := a_0 + a_1 + \cdots + a_n. \tag{4.2.1}$$

(a) In general, "the" continued fraction representation for a rational number q is not unique. (For example, $3 = [3] = [2; 1] = 2 + \frac{1}{1}$. See the Remark following Definition 4.1.1 for additional context about these conventions.) Prove that despite this, $\operatorname{ord}(q)$ *is* uniquely determined by q.

(b) Prove that for every rational number q,

$$\operatorname{ord}\big(\operatorname{left}(q)\big) = \operatorname{ord}\big(\operatorname{right}(q)\big) = \operatorname{ord}(q) + 1. \tag{4.2.2}$$

That is, the order of each child of q is 1 greater than $\operatorname{ord} q$ itself.

4.2.4. Let q be a positive rational with continued fraction expansion $[a_0; a_1, \ldots, a_n]$. In terms of $[a_0; a_1, \ldots, a_n]$, in what row of the Calkin–Wilf tree does q appear? Conversely, if q appears in row m of the tree, can we conclude anything about its continued fraction expansion?

Remark. Much deeper generalizations to Exercise #4.2.4 are possible. That exercise was inspired by Sections 5 and 9 in Bates, Bunder, and Tognetti [3], an article that goes into *much* more detail than we have, posing two ambitious questions:

(a) Let $q > 0$ be any rational number. Knowing q, can we (quickly) determine its location in the Calkin–Wilf tree or the Calkin–Wilf sequence?

(b) Consider the nth entry in the mth row of the Calkin–Wilf tree—*provided* this location makes sense[9] in context. Can we (quickly) determine, in terms of m and n, which rational number q lies at this position in the Calkin–Wilf tree?

Similarly, for an index n in the Calkin–Wilf sequence (c_n), can we (quickly) determine c_n in terms of n?

Remarkably, the answers to the questions in (a) and (b) are both "yes", but pursuing their solutions involves a bit more than we have time for.

Very briefly, those solutions use what [3] calls "normalised additive factors", a way of representing a positive integer as an *alternating* sum of positive and negative powers of 2; for example, $15 = 2^4 + (-1) \cdot 2^1 + 2^0$. You can see some of the relevant patterns by considering the continued fraction expansions for the nth term in *each* mth row. For example, starting in row 3, the 4th entry in each of the following rows are, respectively, $[3] = [2; 1]$ in row 3, $[2; 2] = [2; 2+1]$ in row 4, $[2; 3] = [2; 2+1]$ in row 5, and this pattern continues.

[9] For example, since the second row has precisely two entries, there is no third entry in the second row of the tree.

There are alternate approaches in terms of the Calkin–Wilf sequence that are even simpler to describe. For an index n in the sequence, we can determine c_n using continued fractions, as in [10]. Take the binary/base-two representation of n (as in Exercise #0.3), and look at the lengths of the strings of consecutive 0s and 1s. Those lengths will determine the continued fraction expansion for c_n. Taking the example from [8], we have that $49 = 110001_2$, and we count length of substrings of consecutive digits as $\underbrace{11}_{2}\underbrace{000}_{3}\underbrace{1}_{1}$.

Counting right-to-left, we have 1 consecutive 1s, followed by 3 consecutive 0s, and then 2 consecutive 1s. It follows that

$$c_{49} = [1;3,2] = \frac{9}{7}. \tag{4.2.3}$$

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