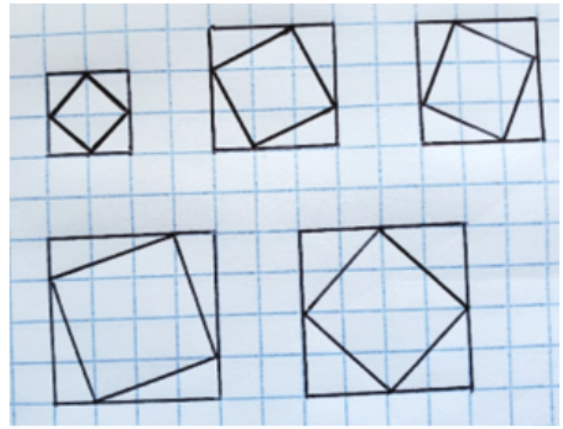
Piece Puzzles

1 - Warm Up



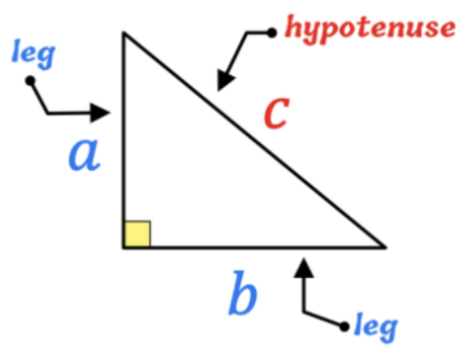
- The area of each grid square is 1.
  - What is the area of each of the inner, "tilted" figures in the picture?
    - Do we need to know the tilted side lengths to answer this?
    - Are there any shortcuts?
  - What shape are these figures?

2 - Pythagorean Puzzle

<u>Terms</u>

- Looking closer at the outer triangles in the picture above, what do they have in common?

We call triangles which contain a *90-degree* angle (like the corner of a book) a special name, *right triangles*.

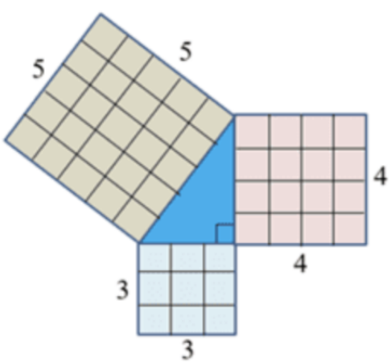


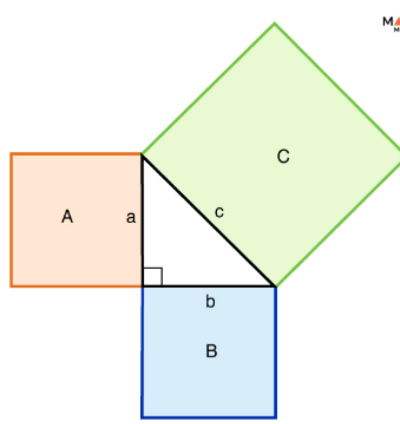
- Here are the parts of a *right triangle* we'll look at:
  - *Hypotenuse (c)*: the longest side (Why longest?)
  - *Legs (a, b)*: the two other sides
- Do you expect the *Hypotenuse (c)* to be bigger, smaller, or the same as the sum of the lengths of the *legs (a+b)*?

<u>Theorem Intro</u>

It turns out that we can relate the 3 side lengths of right triangles in a special way, called the *Pythagorean theorem*. Try this:

- What are the areas of the squares below?



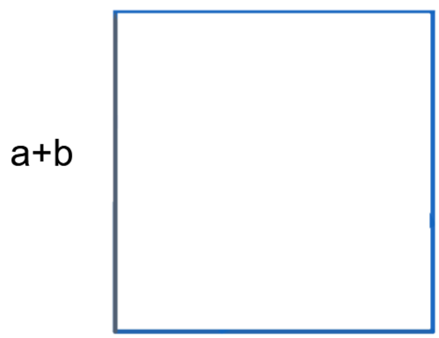


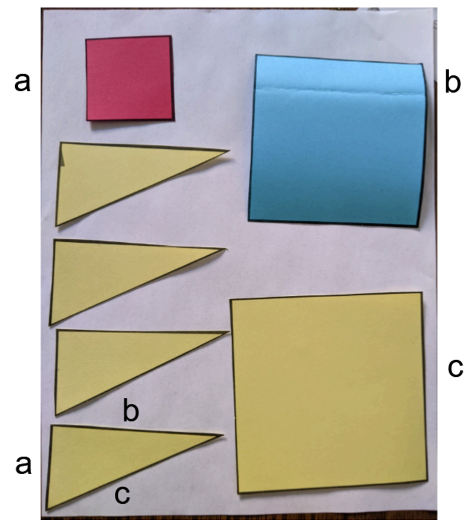
The *Pythagorean Theorem* says that if you square both the smaller sides and then add them up, you get the square of the longest side. We can write this as, $a^2 + b^2 = c^2$. In other words:

- *The combined area of the two smaller squares (formed from the legs) is ________________ to the area of the large square (formed from the hypotenuse)*

Proof

Let's look at a puzzle proof. You'll need a bounding square, as well as: 4 right triangles, a-square, b-square, and c-square.





**See end for cutouts. Instead of letters (a,b,c), you can use Pythagorean triples like: {3, 4, 5} or {5, 12, 13}*

- The goal is to first fill the bounding square with the 4 right triangles and c-square. Draw what you did.

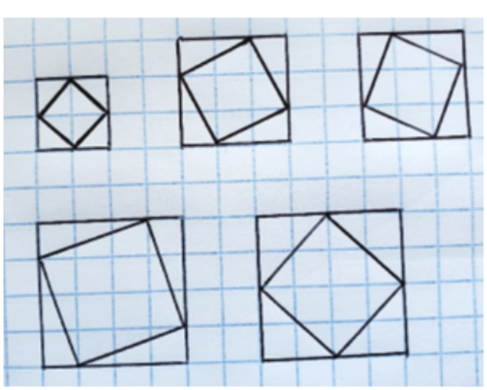
- Now, try to fill the bounding square with the 4 right triangles and both the a-square and b-square. Draw to record this as well.

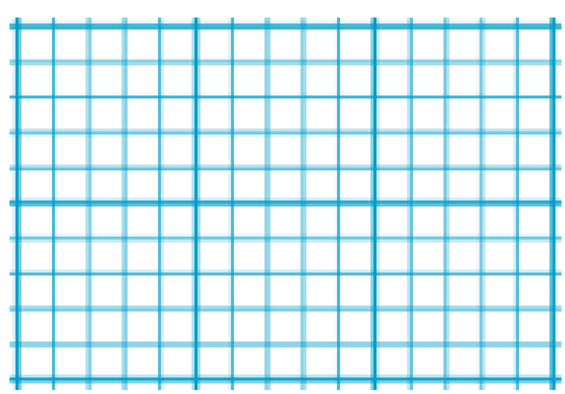
- How does this prove that $a^2 + b^2 = c^2$?

Apply

One immediate application of the *Pythagorean Theorem* is: if we are given two side lengths of a right triangle, we can find the last missing side length.

- Try using the theorem in the outer triangles from the Warm Up above!
- On graph paper, you can also make your own tilted squares (try different sizes) and experiment

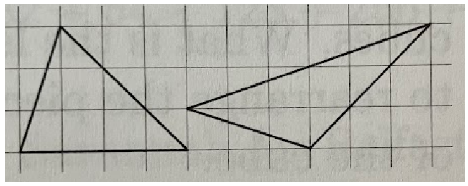




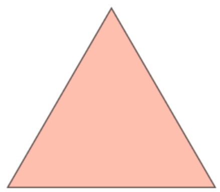
3 - More Puzzles

If there's time, pick and choose some problems below that look fun!

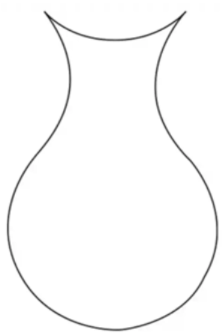
- Cut the first triangle in the diagram into parts that can be reassembled to form the second triangle



- Cut an equilateral triangle into 4 congruent smaller triangles.
    - More broadly, are there triangles (not necessarily equilateral) that you can cut into (a) 3 congruent triangular pieces? (b) 4 congruent triangular pieces? (c) 5 congruent triangular pieces?



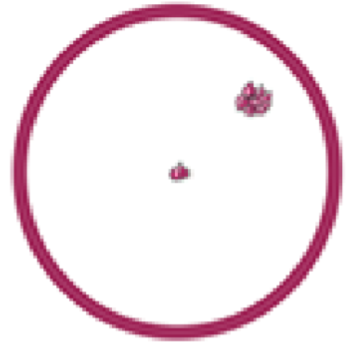
- With 2 straight-line cuts, divide the vase into 3 pieces that can be assembled to form a square.



    - Are there multiple ways to do this?
    - Is there a way to easily express the area of the square?

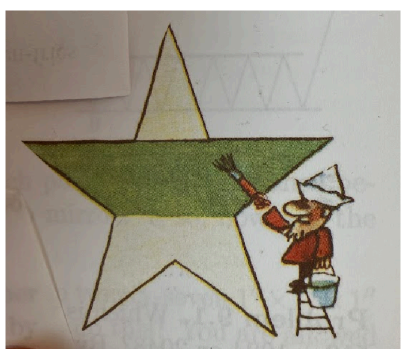
- A page in a calendar is partially covered by the preceding page as in the picture. Which has greater area: the covered part, the uncovered part, or neither?

- A circular cake is decorated with a sugar paste rose as shown in the picture. The center of the circle is marked by a sugar paste dot.
    - Can you cut the cake into 3 pieces so that by rearranging the pieces you create a circular cake again, but this time the rose is in the center?

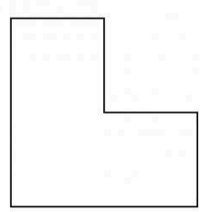


    - Can you cut the cake into just 2 pieces which can be rearranged into a circular cake with the rose in the center? (Note: you can't just remove and replace the rose since it would destroy the icing on the cake.)

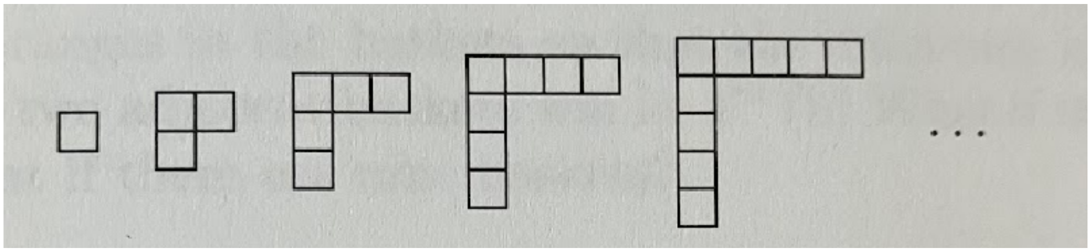
- Show that the area of the green region of the regular pentagonal star in the picture is exactly half of the total area



- Cut the L-shaped figure below into 4 similar figures, each half the size of the original

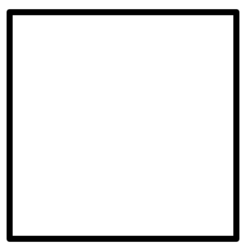


- Here is a series of figures. The first consists of one square.

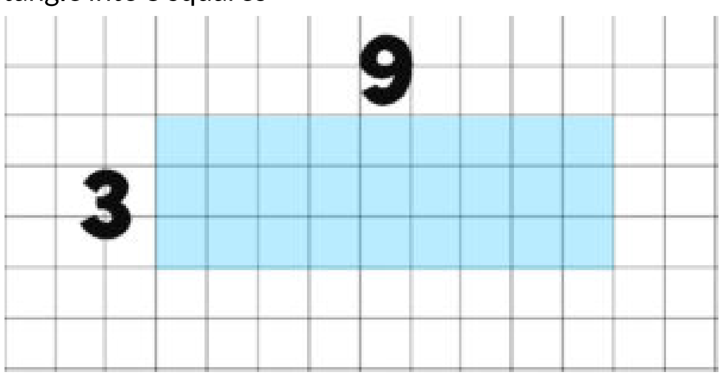


    - How many squares are in the 100th figure?
    - How many squares are in the first 100 figures altogether?

- Cut a square into 5 rectangles in such a way that no two rectangles share a complete common side (but may have some parts of their sides in common).

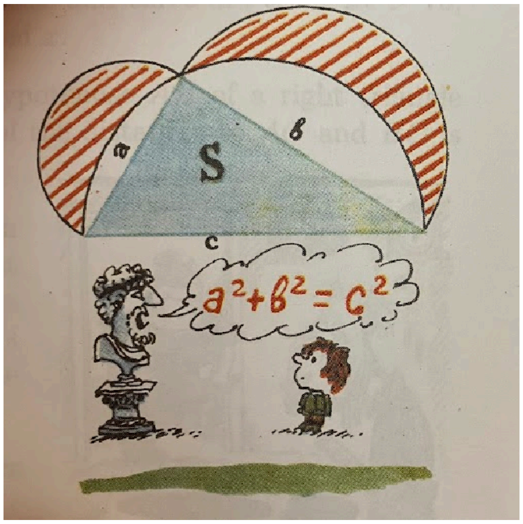


- Cut a 3x9 rectangle into 8 squares

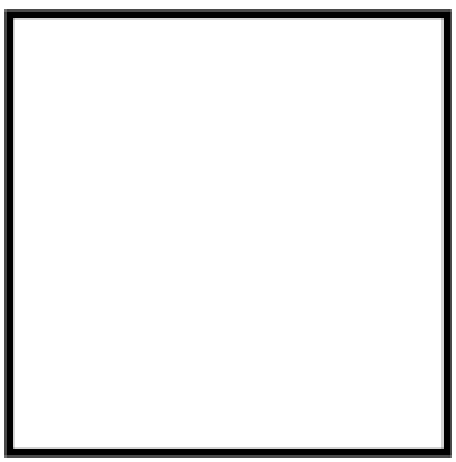


- Cut an arbitrary triangle into 3 parts such that they can be rearranged to form a rectangle

- A right triangle has area S, and its sides(a,b) form diameters of semicircles, as in the diagram above. Find the joint area of the crescents these semicircles form.

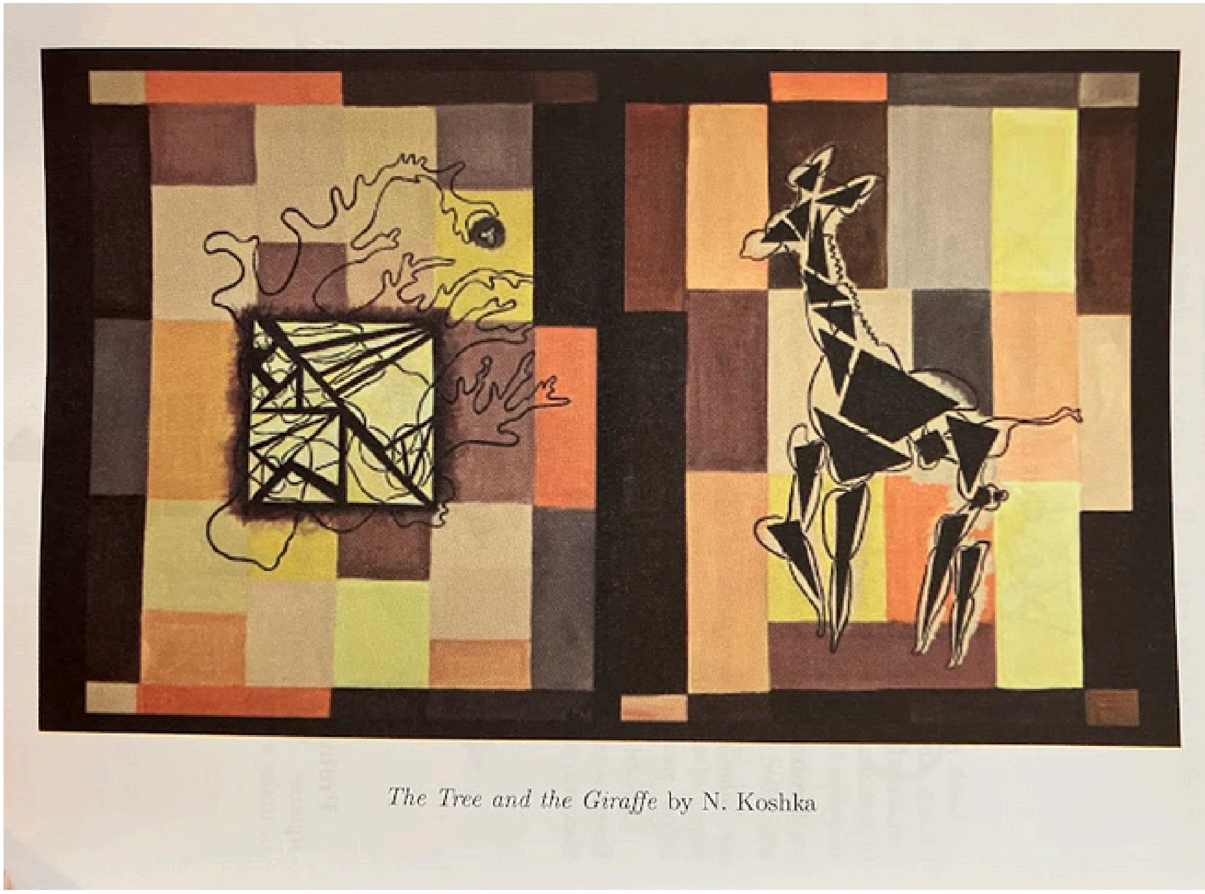


- Which is larger: 1+2+4+8+16 or 32?
- Can you cut a square into 3 smaller squares? What about 4, 5, 6, or 7 smaller squares? (The smaller squares do not have to all be the same size)
    - Are all numbers possible after a certain point?



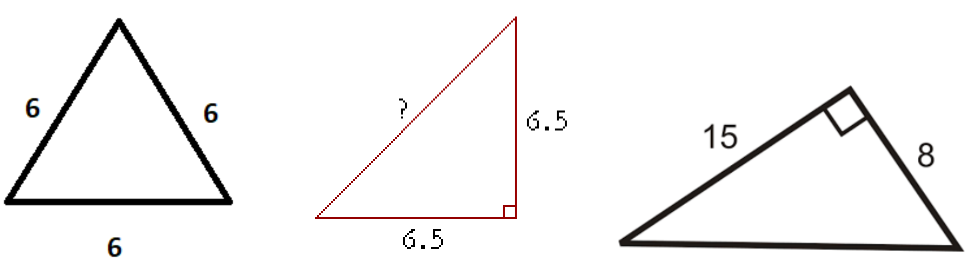
4 - Cut the square

In this painting - *The Tree and the Giraffe (N. Koshka)*- below, what do you notice about the lefthand square and the animal?



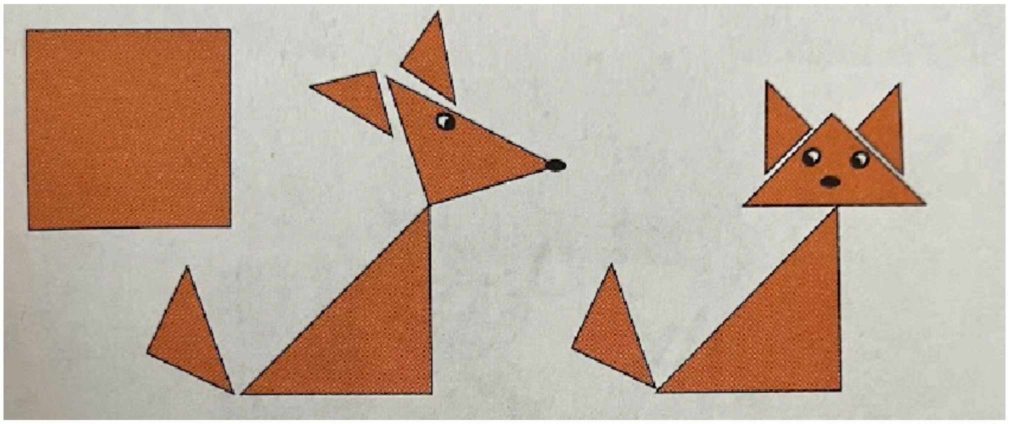
We will use this same idea to make animals out of pieces of squares. But our pieces will only be *squares* or *right isosceles* triangles.

- An *isosceles* triangle has at least 2 sides of equal length. Which of the following triangle(s) are:
  - *right* triangles?
  - *isosceles right* triangles?

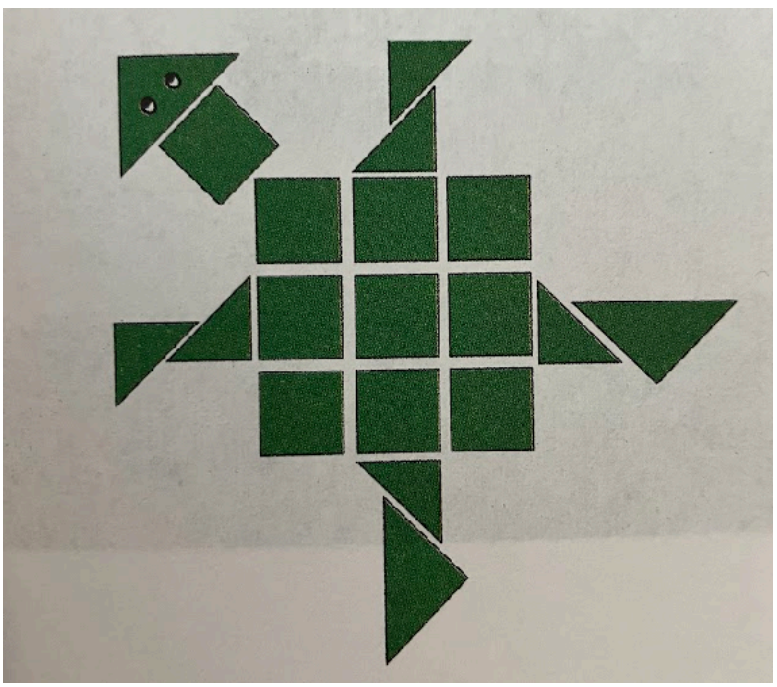


Grab a few pieces of square paper (for the animals), scissors, glue, and a plain paper of another color (as a background surface). You can paste your animals on your plain paper later on when you're done.

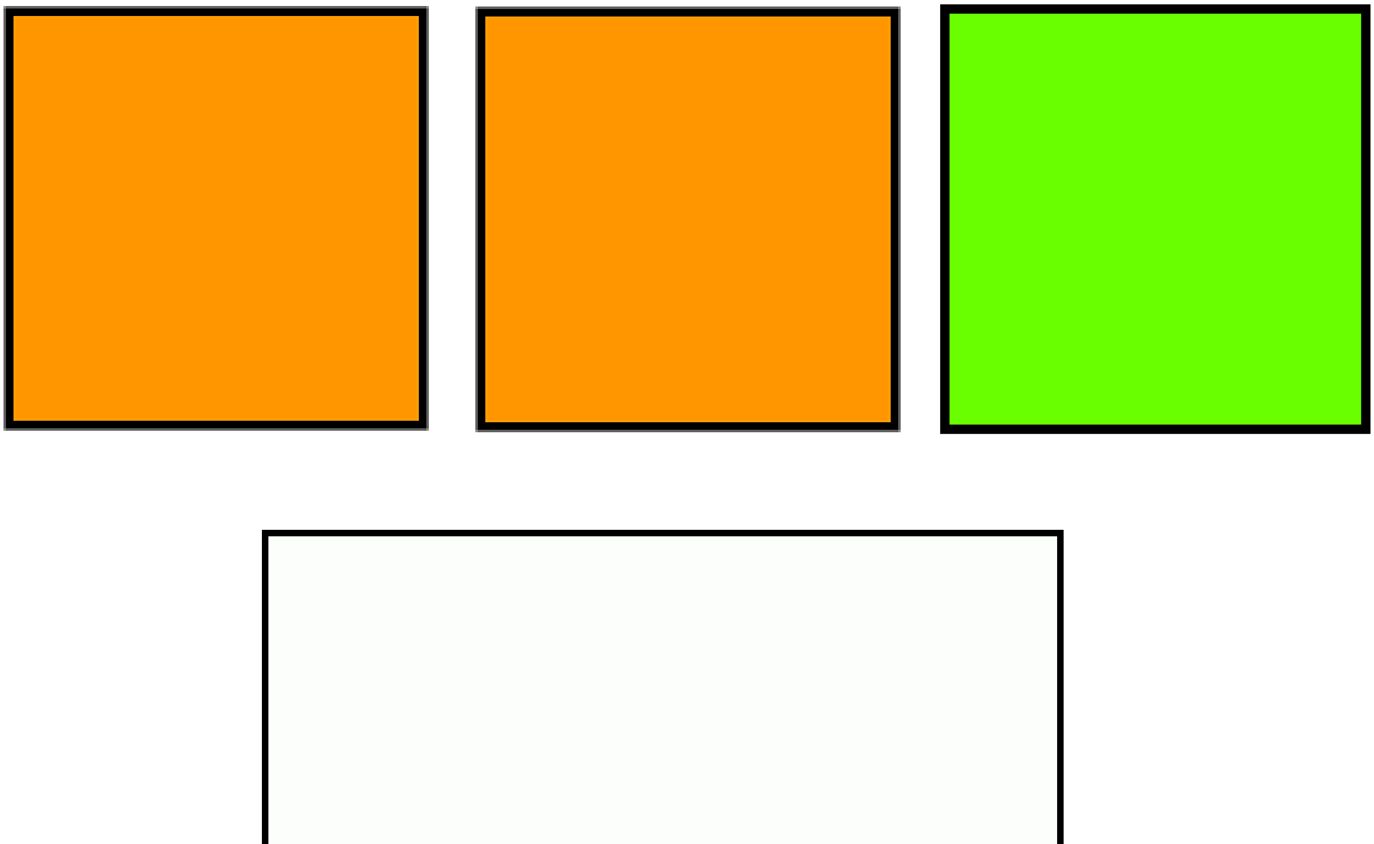
- Cut the square into 5 *right isosceles* triangles (they can be differing sizes), and rearrange them to make the Dog pictured below.
- Can you make the Cat pictured below, using the same pieces?



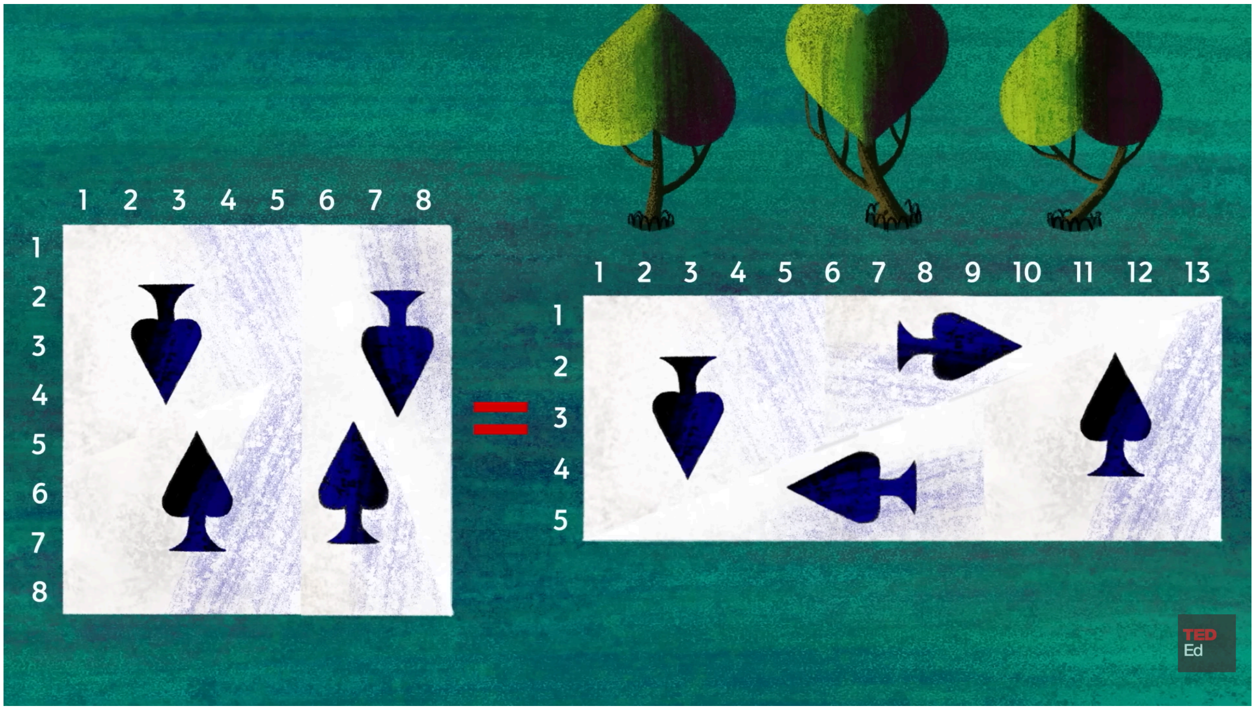
- This Turtle consists of *right isosceles* triangles and *squares*. These are rearranged pieces of a bigger square. Cut a square to make a Turtle.



- Grab a piece of fresh square paper. Can you make your own square animal puzzle? Trade it with another student!



5 - Tricky / False Visuals
TedEd video: [Can you solve the Alice in Wonderland riddle? - Alex Gendler](#) (Fibonacci, illusion)

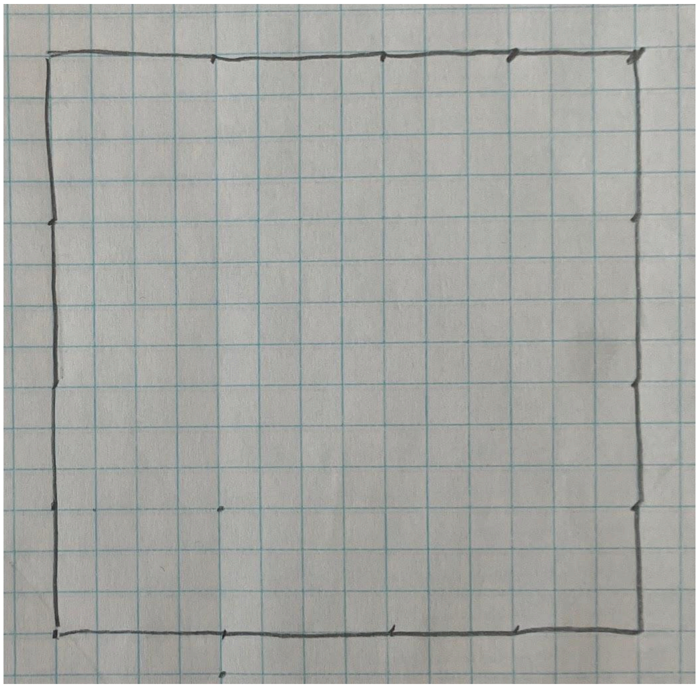


Appendix 1 - Sources

- Fair and Square, Around Pi, Around Pi and Pies (Alliance of Indigenous Math Circles)
- Pythagorean Puzzles (CHMC Spring 2023, Linda Green)
- “Math Circles for Elementary School Student”s (Natasha Rozhkovskaya)
- “A Moscow Math Circle: Week-by-week Problem Sets” (Sergey Dorichenko)
- “Mathematical Circles: Russian Experience” (Dmitri Fomin, Sergey Genkin, Ilia Itenberg)
- See also:
    - Tangrams from https://classplayground.com/tangram/
    - Pentomino Puzzles from https://www.cimt.org.uk/resources/puzzles/pentoes/pentoint.htm

Appendix 2 - Pythagorean Bounding Square and Cutouts

Bounding square (to fill with cutouts)



Cutouts