Chapel Hill Math Circle Exit Ticket:
March 28, 2026: Topics in Topology

Please remove this sheet, complete it, and return it before the end of our session.

- Did you find today's topic interesting?
- Was the this topic appropriately challenging relative to your background? That is, was the topic neither too elementary nor inaccessibly advanced?
- How could we improve this worksheet for future sessions?
- What did you enjoy about today's topic?
- What did you find particularly challenging?
- Was there anything you thought was too difficult?
- Was there anything you thought was too easy?
- Are there any topics you would be interested in seeing us cover in the future?

Topics in Topology

Abstract

Topology is the mathematical study of properties of shapes, *continuous functions* and *continuous deformations*, as well as generalizations of these notions beyond familiar settings of geometry in *Euclidean spaces*. A proper introduction to topology is beyond the scope of this math circle, but in this session we present several topics to introduce concepts and examples from the field. This session will provide introductions to *fractals* and their properties, the *Euler characteristic* and its application to completely classifying all *Platonic solids*, and topological notions sufficient to present *Furstenberg's*[1] *proof that there are infinitely many prime numbers*.

Background needed: Prerequisites include basic algebra and geometry. In particular, it will help to know how to compute the area of an equilateral triangle as a function of the length of its sides, as well as the behavior of *geometric series*: when they converge or diverge, and how to compute the value of its sum when it converges. Additionally, background in introductory set theory will be useful, especially familiarity with the *union* and *intersection* of sets.

Note: Like most Chapel Hill Math Circle worksheets, this document shall be archived at chapelhillmathcircle.org. You can presently find CHMC's archives—and for *all* groups' worksheets, not just those for the advanced group—by navigating to the "Calendar" tab or page, selecting the relevant academic term, then finding the clickable links for each worksheet (where available). Blue text in this document, including in the References, typically provides a clickable link to an external website.

Contents

[1] *Hillel "Harry" Furstenberg (b. 1935)* is an American-Israeli mathematician.

0 Warmup

As prerequisites for this session, it will help to answer the following first. You do *not* need to know these answers already, and many questions will be revisited later in the worksheet.

0.1 What is *mathematical induction*?

Note: Familiarity with mathematical induction will not be strictly necessary for this session, but it may be a useful method for providing rigorous proofs as solutions to some of the exercises below.

0.2 Let S be a set, viewed as a subset of a "universal set" U. What is the *complement of S (in U)*, denoted either $S^{\complement}$ (when U is understood) or $U \setminus X$? If S, T are sets, what is the *union of S and T, denoted $S \cup T$* and the *intersection of S and T, denoted $S \cap T$*? Can you generalize to unions and intersections of more than two sets?

0.3 What is an *arithmetic sequence* (or *arithmetic progression*)? Can you given an example of an arithmetic sequence?

0.4 What is a *geometric series*? If S is the infinite geometric series having initial term a and common ratio r,

$$S := a + ar + ar^2 + ar^3 + \cdots, \tag{0.0.1}$$

when does S converge? When S does converge, what is the value of S?

0.5 What is a *logarithm*? Let a be a positive number with $a \neq 1$. If x is a real number, when is $\log_a x$ defined? When $\log_a x$ is defined, what does $\log_a x$ mean?

0.6 As a subset of the real number, which are denoted $\mathbb{R}$, what is an *interval*? What is an *open interval*? If I, J are open intervals, is the intersection $I \cap J$ an interval? Is it an open interval?

1 Euler Characteristic and the Platonic Solids

1.1 Discussion

In solid geometry, *polyhedra*[2] are a fundamental object of study. In this section, we shall introduce the *Euler*[3] *characteristic* denoted by χ (the lower-case Greek letter chi, pronounced "ky"). As we shall see, the Euler characteristic χ is an *invariant* of all *convex* polyhedra. Using the Euler characteristic and a bit of algebra, we shall completely determine all *Platonic solids*, which are convex, *regular polyhedra* in three dimensions.

We begin with some basic definitions:

[2] *Polyhedra* is the plural for the singular noun *polyhedron*.

[3] Named for the Swiss mathematician *Leonhard Euler (1707–1783)*, who was the protagonist in our exploration of *The Basel Problem* (i.e., computing the sum $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$) in our recent session "π Day 2026" from March 14, 2026. As a reminder: "Euler" is pronounced "oiler".

Definition 1.1.1. A *polygon* P is a two-dimensional shape lying in the plane. Further, P is formed by finitely many straight *edges* which intersect precisely at *vertices*[4], which are endpoints of the edges. Finally, these edges must form a *closed polygonal chain*: if P has n vertices $V_1, V_2, \ldots, V_n$, then the edges $E_1, E_2, \ldots, E_N$ form a loop of the form $E_1 := \overline{V_1V_2}$, $E_2 := \overline{V_2V_3}$, …, $E_{n-1} := \overline{V_{n-1}V_n}$, and $E_n := \overline{V_nV_1}$, where the edges begin and end with the same vertex. A *simple polygon* is a polygon with no self-intersections and no holes.

Remark. In our definition, the polygon itself is simply the one-dimensional set comprising these edges and vertices, *not* the two-dimensional "inside" bounded by those edges and vertices. To distinguish between the one-dimensional polygon and the two-dimensional set determined by a polygon, we shall call the latter a *polygonal region*. Further, we shall typically restrict our attention to simple polygons rather than consider self-intersecting shapes.

Polygons include many familiar shapes: triangles, quadrilaterals, and pentagons, for example. Since polygons are formed by straight edges, though, figures like circles or ellipses are *not* polygons. Let us mention one particular type of polygon:

Definition 1.1.2. A *regular polygon* is a polygon that is *equilateral* (i.e., all sides have equal length) and *equiangular* (i.e., all vertex angles have equal measure).

Remark. Under this definition, a regular polygon may fail to be convex and simple, especially if we allow self-intersections. For example, a *pentagram* is the star-shaped non-simple polygon formed by the diagonals of a regular convex pentagon. Using our definition, the pentagram is a regular polygon. With a relaxed understanding of "equal measure angles", we may also allow simple but non-convex polygons; consider, for example, the white cross within the flag of Switzerland, where all angles are right angles, but some are $270°$ rather than $90°$.

For our purposes, "regular polygon" will mean a regular simple and convex polygon unless indicated otherwise.

Definition 1.1.3. A set S in two- or[5] three-dimensional space is *convex* if and only if for every pair of points $P, Q \in S$, the line segment $\overline{PQ}$ lies entirely inside S.

For example, in Figure 1.1.1, the red shape E is convex because the line segment connecting any pair or red points is also entirely red. However, the green L is non-convex because there exist green points for which the

Most of the sets we shall consider in this section, in both two and three dimensions, will be either convex or the *boundary* of a convex set. For example, a circle is not itself convex, but a closed circular disc is convex, and its boundary is a circle.

[4] *Vertices* is the plural form of *vertex*, the singular noun.

[5] Convexity can be formulated in a more general setting, but sets S that lie in either the plane, $\mathbb{R}^2$, or three-dimensional space, $\mathbb{R}^3$, will be the only examples relevant to today's session.

(a) E: solid ellipse is convex. (b) L: polygonal region is non-convex.

Figure 1.1.1: Illustratinghe defining property of convex sets: the line segment connecting any two red points is red, but this fails in the green region.

Definition 1.1.4. A *polyhedron* is a surface that is the boundary of a three-dimensional shape. It has *faces*, which are polygonal regions, which are themselves bounded by *edges* and *vertices*.

Remark. As with our distinction between the one-dimensional *polygon* being the boundary of its two-dimensional *polygonal region*, we make a similar distinction between a two-dimensional *polyhedron* (or *polyhedral surface*) being the boundary of its three-dimensional *solid polyhedron* (or *polyhedral solid*). We also make analogous distinctions as with polygons: a *simple polyhedron* is one for which all its faces are simple polygons and for which the solid polyhedron has no nontrivial self-intersections. Similarly, a *convex polyhedron* is one where for any two points in the solid polyhedron, the line segment connecting those points lies entirely inside the solid polyhedron.

Unless otherwise specified, you may assume any mention of "polyhedron" refers to a simple and convex polyhedron unless otherwise specified.

Definition 1.1.5. A *regular polyhedron* is a polyhedron such that every face is the polygonal region of a regular polygon, the faces are all mutually congruent as polygonal regions, and faces sharing a common edge meet at the same *dihedral angle*.

Definition 1.1.6. A *Platonic*[6] *solid* is the solid polyhedron for a a simple, convex regular polyhedron. We call the polyhedral boundary to the Platonic solid a *Platonic boundary*.

Remark. In a regular polyhedron, we require that each face is the *same* regular polygon. To elaborate on why, a *truncated icosahedron*, a polyhedron approximated by a soccer ball (as in Figure 1.1.2), is such that each face is a regular polygon. However, 20 of its 32 faces are regular hexagons, and the remaining 12 are regular pentagons. This disqualifies the truncated icosahedron from qualifying as a regular polyhedron because the faces are not all the same. (Even if this weren't by itself disqualifying, the dihedral angles between neighboring hexagons is approximately 138.18°, while the angle between adjacent hexagonal and pentagonal faces is approximately 142.6°.)

[6]Named for *Plato (born c. 428–423 BCE, died 348 or 347 BCE)*, a philosopher from ancient Athens.

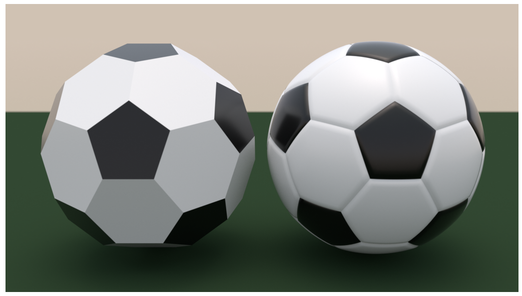


Figure 1.1.2: Comparing a truncated icosahedron (left) to a soccer ball. Taken from https://upload.wikimedia.org/wikipedia/commons/0/08/Comparison_of_truncated_icosahedron_and_soccer_ball.png by Aaron Rotenberg, *Wikipedia*, August 23, 2014. Used under Creative Commons Attribution-Share Alike 4.0 International license. Original image is unedited, though resized to fit this document.

Example 1.1.7. The following are examples of Platonic solids (using LaTeX and Ti*k*Z code modified from Sebastiano Tronto's GitHub archive):

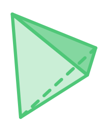


Figure 1.1.3: The regular tetrahedron.

Remark. These Platonic solids may be familiar to those of you who've played tabletop games like *Dungeons & Dragons* as the shapes of several of the dice. The shapes above correspond to, respectively, "D4", "D6", "D8", "D12", and "D20", where "Dn" denotes a dice shaped like a Platonic solid with n faces.

Definition 1.1.8 (Euler characteristic)**.** Let P be a (simple convex) polyhedron having V vertices, E edges, and F faces. Then the *Euler characteristic* of P, denoted χ or $\chi(P)$, is the value

$$\chi := V - E + F. \tag{1.1.1}$$

Example 1.1.9. Let O be the regular octahedron in Figure 1.1.5. Then for this O, we have

$$V = 6,$$
$$E = 12, \text{ and}$$
$$F = 8.$$

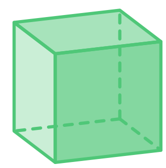


Figure 1.1.4: The cube.

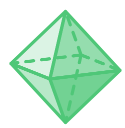


Figure 1.1.5: The regular octahedron.

Combining this with the definition of χ in Equation (1.1.1), we see that for O,

$$\boxed{\chi(O) = 2.} \quad (1.1.2)$$

Remark. The Euler characteristic can be generalize to a context of *planar graphs* or shapes other than simple, convex polyhedra. For our purposes here, we shall consider the Euler characteristic solely in the context of simple, convex polyhedra. For a more general exploration of the Euler characteristic of surfaces, including non-orientable surfaces like the *Möbius*[7] *strip* or the *Klein*[8] *bottle*, see our session "Topology of Surfaces and the Euler Characteristic" from April 12, 2025.

1.2 Exercises

1.2.1 In Figure 1.2.1, which of these sets in the plane are convex, and which are non-convex? Can you produce examples of convex and non-convex sets in three-dimensional space?

[7] Named for *August Ferdinand Möbius (1790–1868*, a German mathematician and theoretical astronomer. (He is also the namesake for the Möbius Noodles group here at Chapel Hill Math Circle.)

[8] Named for *Felix Klein (1849–1925)*, a German mathematician, educator, and historian of mathematics.

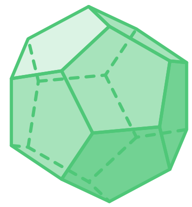


Figure 1.1.6: The regular dodecahedron.

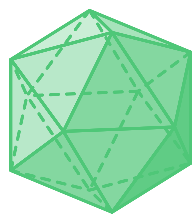


Figure 1.1.7: The regular icosahedron.

1.2.2 Complete the Table 1.2.1 to calculate the Euler characteristic χ for the given simple, convex polyhedra. Next, add some of your own examples of simple, convex polyhedra, and compute χ for those. What patterns do you notice?

Note: Computing V, E, and F for the truncated icosahedron will likely be most challenging among these examples.

What patterns do you notice? Do you have any conjectures? Can you prove them?

1.2.3 To simplify working with three-dimensional solid polyhedra and their two-dimensional boundaries, let us explore a simplification. Specifically, we shall consider *Schlegel*[9]

[9]Named for *Victor Schlegel (1843–1905)*, a German mathematician.

(a) P: polygon and interior. (b) K: polygon and interior. (c) D: closed disc. (d) A: doubly-open annulus.

Figure 1.2.1: Examples of convex and non-convex sets. Which are convex, and which are non-convex?

diagrams of simple convex polyhedra as a way to simplify studying a polyhedron in space by instead working with simpler objects in the plane.

(a) Consider the regular tetrahedron in Figure 1.1.3. If we project the tetrahedron onto the plane, stretching edges without tearing or intersecting them, we can obtain the following:

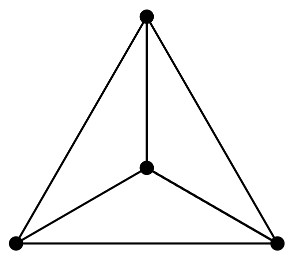


Figure 1.2.2: A Schlegel diagram for a regular tetrahedron.

Viewing Figure 1.2.2 as a graph (in the sense of *graph theory*, how many vertices are there? How many edges? How many faces, counting the entire "outside" region as a face, too? How does this compare to your answer to the corresponding entry in Table 1.2.1 as your answer to Exercise #1.2.2?

(b) Consider the cube in Figure 1.1.4. If we again project the cube onto the plane, stretching edges without tearing or intersecting them, we can obtain the following:

In Figure 1.2.3, how many vertices appear? How many edges? How many faces, counting the entire "outside" region as a face? How does this compare

polyhedron	V	E	F	$\chi := V - E + F$
regular tetrahedron (Figure 1.1.3)	4	6	4	2
pyramid	5	8	5	2
cube (Figure 1.1.4)	8	12	6	2
prism whose base is a regular n-gon				
regular octahedron (Figure 1.1.5)				
regular dodecahedron (Figure 1.1.6)				
regular icosahedron (Figure 1.1.7)				
truncated icosahedron (i.e., soccer ball)				
my polyhedron #1				
my polyhedron #2				
my polyhedron #3				
my polyhedron #4				
my polyhedron #5				
⋮	⋮	⋮	⋮	⋮

Table 1.2.1: Computing the Euler characteristic χ for various simple convex polyhedra.

to your answer to the corresponding entry in Table 1.2.1 as your answer to Exercise #1.2.2?

(c) Repeat this process of forming a Schlegel diagram for another simple convex polyhedron, such as one of those in Table 1.2.1.

(d) Using Schlegel diagrams, explain how if P is any simple convex polyhedron, then its Schlegel diagram will be a *planar graph*. That is, the vertices, edges, and faces of the Schlegel diagram can be embedded in the plane so that the edges intersect only at their endpoints, which are vertices. Furthermore, including the "infinite face" in the Schlegel diagram, we have that for any Schlegel

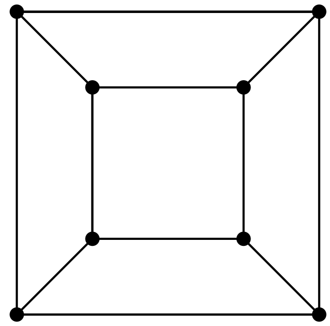


Figure 1.2.3: A Schlegel diagram for a cube.

diagram arising from a simple convex polyhedron, the value $V - E + F$ is the same, whether viewed in the original polyhedron or in its Schlegel diagram.
Note: It follows that to compute $\chi = V - E + F$ in a simple convex polyhedron, it suffices to compute this value for its associated Schlegel diagram.

1.2.4 Your results from Table 1.2.1 should suggest the following:

Conjecture 1.2.1. *Let P be any simple, convex polyhedron. Then*

$$\chi(P) = 2, \tag{1.2.1}$$

where $\chi(P)$ denotes the Euler characteristic of P. Explicitly, for any such P, and with V, E, and F denoting, respectively, the number of vertices, edges, and faces to P.

$$V - E + F = 2. \tag{1.2.2}$$

Potentially Challenging: Prove Conjecture 1.2.1.

Note: There are many different methods for proving that $\chi(P) = 2$ for any simple convex polyhedron. For example, *The Geometry Junkyard* by David Eppstein presents "Twenty-one Proofs of Euler's Formula: $V - E + F = 2$", though a single valid proof will suffice for our purposes.

1.2.5 In Example 1.1.7, and Figures 1.1.3–1.1.7, we enumerated five Platonic solids: the regular tetrahedron, the cube, the regular octahedron, the regular dodecahedron, and the regular icosahedron. Using the Euler characteristic, this exercise will outline a method of proving that these are the *only* Platonic solids.

Let P be any Platonic solid. Assume that each face of P is the polygonal region of a simple, convex regular n-gon; that is, we assume that each face in P has n edges and n vertices. Further, let k denote the number of faces that share any given vertex; since, by hypothesis, P is a Platonic solid, this number k must be identical for each vertex.

(a) With n, k as above, explain why we must have

$$n \geq 3, \text{ and} \tag{1.2.3}$$

$$k \geq 3. \tag{1.2.4}$$

(b) Show that

$$nF = 2E. \tag{1.2.5}$$

Conclude that

$$F = \frac{2E}{n}. \tag{1.2.6}$$

Hint: Every edge is shared by precisely 2 faces, and each face has n edges. Taking potential double-counting into account, what can you conclude?

(c) Show that

$$kV = 2E. \tag{1.2.7}$$

Deduce that

$$V = \frac{2E}{k}. \tag{1.2.8}$$

(d) From Exercises #1.2.5(b)–1.2.5(c) and our formula in Equation (1.2.2), explain why

$$\frac{2E}{n} - E + \frac{2E}{k} = 2. \tag{1.2.9}$$

Since $E > 0$, conclude that

$$\frac{1}{n} + \frac{1}{k} > \frac{1}{2}. \tag{1.2.10}$$

(e) Recall that n, k are *integers*. Combining this with Equations (1.2.3), (1.2.4), and (1.2.10), explain how the only candidate solutions (n, k) are

$$(n, k) = (3,3), (4,3), (3,4), (5,3), \text{ or } (3,5). \tag{1.2.11}$$

(f) Show each candidate solution in Equation (1.2.11) is indeed one of the Platonic solids enumerated in Example 1.1.7. Conclude that the list there gives *every* Platonic solid.

Remark. There are other approached to prove that these are the only Platonic solids. For example, another approach uses the notion of an *angle defect*.

Separately, we can generalize the Euler characteristic to higher dimensions. Now, though, rather than simply considering vertices, edges, and two-dimensional faces, we must also include three-dimensional "hyperfaces", and even higher-dimensional extensions depending on the dimension of an n-dimensional simple convex polyhedron.

2 Fractals

2.1 Discussion

In introductory mathematics classes, we typically begin by studying simple shapes like polygons, circles, ellipses, polyhedra, spheres, and other similar shapes. There are many shapes, though, which have seemingly pathological properties, such as having a dimension that is not an integer! These sets are called *fractals*, and we shall explore their interesting—and counterintuitive—properties. Fractal-like patterns are observed everywhere in the physical world: networks of blood vessels in a body, the surface of waves, the apparent roughness of the contour of a coastline, no matter the level of resolution, and the intricacy of snowflakes. Fractals are useful, too, in terms of modeling real-world phenomena, designing types of antennae, and even making early medical diagnoses.

Caveat 2.1.1. A truly rigorous definition of fractals shall remain beyond the scope of this session for a few unavoidable reasons. First, the background needed to present such a definition, let alone understand it, would require more theoretical background than we can present here. A more fundamental reason is that appears to be no consensus on a single definition of "fractal". Despite this, we hope that the examples below shall illuminate many key properties of canonical examples of fractals.

Example 2.1.2 (Cantor Set)**.** Let C_0 be the interval $[0,1]$. Passing from C_0 to C_1, we remove the open middle third interval, $\left(\frac{1}{3},\frac{2}{3}\right)$. Passing again from C_1 to C_2, we remove the open middle thirds of each component interval of C_1.

Continuing in this way, for every nonnegative integer n, we can form C_{n+1} by removing the middle open third interval from each component interval of C_n. The *Cantor*[10] *set*, here denoted by C, is the set of points lying in *every* set C_n. See Figure 2.1.1.

Example 2.1.3 (Koch Snowflake)**.** Let K_0 be an equilateral triangle. Passing to K_1, at each side of K_0, replace the middle third by line segments as indicated in Figure 2.1.3. Passing from K_1 to K_2, we remove the open middle quarter equilateral triangles from each of these triangles.

Continuing in this way, for every nonnegative integer n, we can form K_{n+1}. The set K of points lying on the boundary of this snowflake-like regions region is called the *Koch*[11] *snowflake*. See Figures 2.1.2 and 2.1.3.

These and many other examples of fractals exhibit *self-similarity*. This is a common property of fractals, but as explained in [2], self-similarity is not required in the definition of a fractal.

[10]Named for *Georg Cantor (1845–1918)*, a European mathematician who made foundational contributions to set theory, especially in his study of infinite sets.

[11]Named for *Helge von Koch (1870–1924)*, a Swedish mathematician.

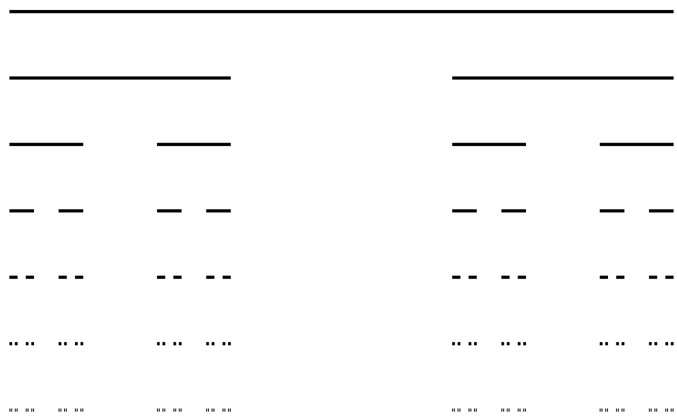
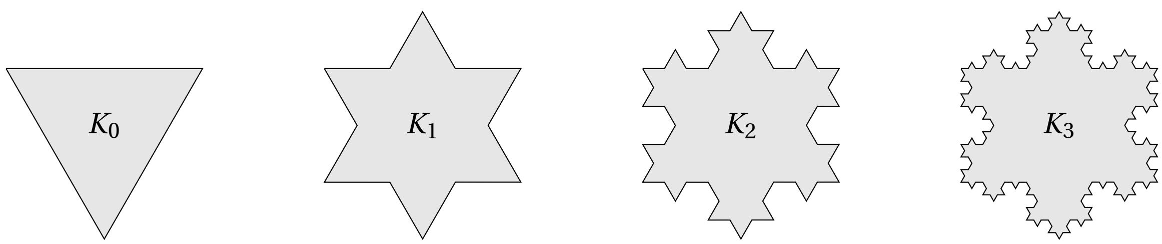


Figure 2.1.1: Forming the Cantor set C from the sequence (C_n).



(a) K_0: an equilateral triangle, together with its bounded region.

(b) K_1: first iteration.

(c) K_2: second iteration.

(d) K_3: third iteration.

Figure 2.1.2: Constructing the Koch snowflake fractal.

2.2 Exercises

2.2.1 Assume that the side length of the original equilateral triangle in Figure 2.1.2a is 1 (or, if you prefer more generally, s). What is the perimeter of K_0, and what is the area enclosed by K_0? What are the perimeter and area of K_1? Of K_2? What do you conclude about the perimeter—to the extent "perimeter" is meaningful to measure here—and enclosed area of K?

2.2.2 In the iterative construction of the hat is the length of $C_0 = [0,1]$? What is the overall

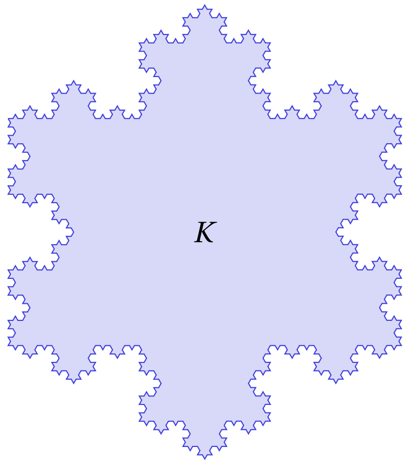


Figure 2.1.3: Approximation to the Koch snowflake, K.

length of the remaining components of C_1? Of C_2? What does this suggest about the length of C, the Cantor set itself?

2.2.3 Exercise #2.2.2 means, intuitively, that the Cantor set C is very small as a subset of $\mathbb{R}$. Show that despite this, C is an infinite set.

Challenging: Can you show that C is not only infinite but also uncountably infinite?

2.2.4 If S is a square and I scale it up linearly by a factor $k > 0$, by what factor the area of S change? If C is a cube, and I scale it up linearly by a factor of k, by what factor does its volume change?

2.2.5 In Exercise 2.2.4, note that the exponents 2 and 3 in our answer correspond to the respective *dimensions* of the square and cube. That is, In these cases, the dimension d satisfies

$$\text{scaling the region by a factor of } d \text{ scales its area/volume/mass by a factor of } k^d. \tag{2.2.1}$$

With that as motivation, consider the Cantor set, C. If I scale up C by a factor of 3, we get *two* copies of C. This suggest that if d is the dimension of C, then d satisfies the equation

$$3^d = 2. \tag{2.2.2}$$

Use this to compute $d = \dim C$. (Here, d is typically called the *fractal dimension* or sometimes *similarity dimension* for C.)

2.2.6 In the Koch snowflake K, at each iteration, we replace a line segment of length s with one of length $\frac{4}{3} \cdot s$. Explain why this means that when d denotes the fractal dimension of K, d satisfies the equation

$$3^d = 4. \tag{2.2.3}$$

Use this to find a closed-form expression for d.

3 An Introduction to Point Set Topology and Elementary Number Theory

Topology can be (informally) described as the study of properties of “spaces", which can be thought of as sets with some extra structure (we'll make this more precise later). When we encounter the definition of a topology for the first time, it can seem unmotivated, but familiar examples might help us to see where the properties of a topology come from.

3.1 Open Intervals, Open Sets, and Topology

Consider an open interval in the set of real numbers, e.g., $(0,1) = \{x : 0 < x < 1\}$. For any number x that's between 0 and 1, we have the inequalities

$$0 < \frac{x}{2} < x < \frac{x+1}{2} < 1$$

This means that for any real number x that's strictly between 0 and 1, we can "move around" x for any $0 < x < 1$ and still stay in the interval $(0,1)$, since there will always be numbers less than x and greater than x in the interval. This is an example of what we'll later call an **open** set. Now let's look at the union of two open intervals, for example $(0,1)\cup(1,2)$. We still have the same property of being able to "move around" any number in our union of open intervals. We can also show that we can take the union of any number of open intervals, and we can still "move around" in this union of open intervals (If we can "move around" x for x in an open interval, we can still move around x if we add more open intervals to the one we started with.)

However, taking intersections is a little more restrictive. Consider two "open" sets $(0,1)$ and $(0.5,1.5)$. Their intersection is $(0.5,1)$, another "open set" by our intuitive definition above. In fact, the intersection of finitely many such sets is also "open", but this is not true for infinite intersections. Consider the sets $A_n = (-\frac{1}{n}, 1+\frac{1}{n})$. The intersection $\bigcap_{n=1}^{\infty} A_n = [0,1]$, which is not open (can you see why?)

These two observations are what we aim to mimic in the definition of a topology.

Definition 3.1.1. A topology on a set X is a collection $\mathcal{T}$ of subsets of X so that

1. $\varnothing \in \mathcal{T}$, $X \in \mathcal{T}$.
2. If $(U_i)_{i\in I} \subset \mathcal{T}$ then $\bigcup_{i\in I} U_i \in \mathcal{T}$.
3. If U_1, $U_2 \in \mathcal{T}$ then $U_1 \cap U_2 \in \mathcal{T}$.

We'll call the sets in $\mathcal{T}$ **open**. If a set V is of the form $V = X\setminus U$ with $U \in \mathcal{T}$, we will say that V is **closed**. It is possible for subsets of X to be neither open nor closed, and it is also possible for sets to be open and closed. Here are some examples of topologies

1. $X = \mathbb{R}$, $\mathcal{T} = \mathcal{P}(X)$, i.e $\mathcal{T}$ contains every subset of X. In this topology, every subset of X is both open and closed. This is called the *discrete* topology.
2. $X = \mathbb{R}$, $\mathcal{T} = \{\varnothing, X\}$. Note that the set $\{1\}$ is neither open nor closed in this topology. This is called the *indiscrete* topology.

For the rest of this document, when we say space, we really mean a set with a topology.

Exercise: Can you see why the discrete and indiscrete topologies collections are always topologies on any set X? Try to prove this.

From the properties of a topology, we can say that any union of open sets is open, and the intersection of two open sets is open. By induction, the intersection of finitely many open sets is open. We can also say analogous things about closed sets: The union of finitely many closed sets is closed, and any intersection of closed sets is closed.

Working through these definitions can feel tedious initially, but topology can be an extremely powerful tool and can be applied to prove strong results in virtually every subfield of math, e.g., the fundamental theorem of algebra. One can prove using topology that if we crumple a world map and drop it anywhere on the Earth, there is at least one point on the paper that lines up with the point on the Earth that it represents. Another theorem allows us to prove that there is at least one point on the earth that has the same temperature and atmospheric pressure as its “antipodal" point (the opposite point on the globe).

3.2 Properties of Primes

Today, we'll go through two proofs that the set of primes is infinite. The first one is due to Euclid, and the second one (due to Furstenberg) is much more recent, and involves a slick application of topology.

Theorem 3.2.1. *There are infinitely many prime numbers.*

Before reproducing Euclid's proof, we'll show a preliminary result.

Theorem 3.2.2. *Every natural number can be factored into a product of primes.*

Proof. We'll prove this statement by strong induction.

- <u>Base case:</u> $n = 1$ is a product of 0 primes, hence is a product of primes.

- Inductive Hypothesis: Assume that every natural number $n \leq k$ has a prime factorization.
- Inductive step **(Exercise)**: There are two possibilities for the number $n+1$, it is either prime or composite. Try to prove the inductive step in each case (Hint: If $n+1$ is composite, then $ab = n+1$ for some a and b. What can we say about a and b?)

□

We will reproduce Euclid's proof here. The key step behind Euclid's proof is showing that no finite set of primes can be the set of all prime numbers.

Proof of Theorem 3.2.1 (Euclid). Assume, instead, that there are finitely many prime numbers:

- Let $\{p_1, \ldots, p_n\}$ be a finite set of prime numbers.
- **(Exercise)** Recall that every natural number can be factored into a product of primes. This means that every natural number has at least one prime number that divides it. Find a number that has a prime factor different from $p_1, \ldots, p_n$. Note that we don't need this new number to be prime
(Hint: What is a number that has all the listed prime numbers as prime factors?)
(Extra hint: Play around and experiment with common factors of numbers of the form n and $n+k$ for different values of k. For example, what are the possible common factors of n and $n+2$ for different values of n? What about n and $n+3$? n and $n+5$?)
- Deduce that if there were only finitely many prime numbers, then we would arrive at a contradiction. Conclude that there must be infinitely many primes.

□

4 Connectedness

We call a space X **disconnected** if there are two nonempty open sets U and V so that $X = U \cup V$, and $U \cap V = \varnothing$, i.e we can find nonempty open sets U and V so that every element of X is either in U or in V, and U and V have no common elements. A space X is called **connected** if it is not disconnected.
Here are two examples:

Let $X = \{1,2\}$ with the topology $\mathcal{T} = \{\{1\},\{2\},X,\varnothing\}$. Then X is disconnected.

Let $X = \{1,2,3\}$ with the topology $\mathcal{T} = \{\varnothing,\{1\},\{1,2\},X\}$. Then X is connected.

Exercise: Prove the claims above, or try to think about why they are true.

Exercise: Give some examples of disconncted/connected spaces and prove that they are disconnected/connected.

Exercise: Is it possible to put a topology on a set that always makes it connected? What about a topology that always makes it disconnected?

5 Furstenberg's Proof of the Infinitude of Primes

This will be a proof of the claim that the set of prime numbers is infinite, and the claim will be proved using a topology on the set of integers $\mathbb{Z}$.

The topology: Let $\mathfrak{T}$ be the topology on the set $\mathbb{Z}$ that consists of sets U that can be written as a union of arithmetic progressions, i.e we will call a set U open if U can be written as a union of sets of the form $\{\ldots, a-2b, a-b, a, a+b, a+2b, \ldots\}$ where a and b are integers. We'll refer to the set $\{\ldots, a-2b, a-b, a, a+b, a+2b, \ldots\}$ by $S(a, b)$.

Exercise 1: Can you see why this is a topology? Try to prove this.

Exercise 2: Can you give an example of a nonempty finite open set? If not, can you say why there are no finite open sets?

Exercise 3: Can you show that $S(a, b)$ is closed (recall that a closed set is defined as the complement of an open set)? (Hint: Try to show that we can write $S(a, b)$ as $S(a', b)$ for some integer $0 \leq a' < b$. Think of $S(a,b)$ as the set of integers that have remainder a' when divided by b, and use the fact that any number when divided by b has only finitely many possible remainders.)

Exercise 4: Prove that we can write $\mathbb{Z} \backslash \{1,-1\} = \cup_{p \text{ prime}} S(0,p)$. That is, any integer n is divisible by *some* prime, with the exception of $n = \pm 1$.

Exercise 5: Is the left hand side i.e the set $\mathbb{Z} \backslash \{1,-1\}$ closed/not closed/open/not open? (Keep in mind that saying "not closed" is not the same as saying "open"!) Prove your claim. (Hint: Recall Exercise 2).

Exercise 6: Assume that there are finitely many primes: Is the right hand side i.e the set $\cup_{p \text{ prime}} S(0,p)$ open/closed/not closed/not open? Prove your claim. (Hint: Recall the properties of closed sets, and Exercise 3)

Exercise 7: If we assume that there are really finitely many primes, show that your conclusions from Exercise 5 and 6 will lead you to a contradiction.

References

[1] 3Blue1Brown. Fractal charm: Space filling curves. https://www.youtube.com/watch?v=RU0wScIj36o, January 16, 2016.

[2] 3Blue1Brown. Fractals are typically not self-similar. https://www.youtube.com/watch?v=gB9n2gHsHN4, January 27, 2017.

[3] 3Blue1Brown. Who cares about topology? (Old version). https://www.youtube.com/watch?v=AmgkSdhK4K8, November 4, 2016.

[4] Keith Conrad. The "topological" proof of the infinitude of primes. https://kconrad.math.uconn.edu/blurbs/ugradnumthy/primestopology.pdf, April 30, 2019. online: retrieved March 25, 2026.

[5] Gerald Edgar. *Measure, Topology, and Fractal Geometry*. Undergraduate Texts in Mathematics. Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA, 1990.