CHAPEL HILL MATH CIRCLE EXIT TICKET:
March 14, 2026: π Day 2026

Please remove this sheet, complete it, and return it before the end of our session.

- Did you find today's topic interesting?

- Was the this topic appropriately challenging relative to your background? That is, was the topic neither too elementary nor inaccessibly advanced?

- How could we improve this worksheet for future sessions?

- What did you enjoy about today's topic?

- What did you find particularly challenging?

- Was there anything you thought was too difficult?

- Was there anything you thought was too easy?

- Are there any topics you would be interested in seeing us cover in the future?

π Day 2026

Abstract

Today, March 14, is *Pi Day*, founded in 1988 as a celebration of the mathematical constant $\pi = 3.14159...$. Our topic for this session will explore certain properties and unexpected appearances of π. We begin by considering geometric methods for approximating π, comparing the circumference of a circle of radius 1 to the perimeters of inscribed regular polygons. We then consider *Buffon's Needle Problem*, where π arises in computing the probability that a randomly dropped needle intersect with parallel lines separated by a common distance. This section will introduce, in a conceptual way, core ideas from From there, we proceed to *The Basel Problem*, where π arises when computing the sum $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$. We conclude by extending techniques from our solution to the Basel Problem to introduce the *Riemann zeta function*, which we then use to compute the probability that a randomly selected set of finitely many positive integers is relatively prime.

Background needed: Prerequisites include basic algebra and geometry. Sections 1–3 will use some results from trigonometry. Section 2 will also introduce concepts from *integral calculus*, an introduction to methods for computing areas of general regions, but without presupposing prior background with calculus. Sections 3–4 will consider *infinite series* (i.e., sums of infinitely many terms) and *infinite products*, but with an intuitive rather than a rigorously technical approach. Section 4 will use certain results from number theory, such as properties of primes and criteria for determining whether two integers a, b are relatively prime. Finally, Sections 2 and 4 will use notation for probability. (For example, $P(\theta < x)$ denotes the probability that θ is less than x, and $P(\gcd(a, b) = 1 : a, b, \in \mathbb{N})$ denotes the probability that the integers a and b, randomly chosen from the set $\mathbb{N} := \{1, 2, 3, \ldots\}$, are relatively prime.)

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Contents

0 Warmup

As prerequisites for this session, it will help to answer the following first. You do *not* need to know these answers already, and many questions will be revisited later in the worksheet.

0.1 How might you define *the mathematical constant π*? What is the approximate value of π? Do you know any properties[1] of π? What interesting properties or equations involving π can you remember?

0.2 What is *(capital) sigma notation* to denote sums? For example, what do the sums

$$\sum_{k=1}^{n} a_k, \sum_{k=j}^{n} a_k, \text{ and } \sum_{k=1}^{\infty} a_k \tag{0.0.1}$$

mean?

Similarly, what is *(capital) pi notation* for products? What do the products

$$\prod_{k=1}^{n} a_k, \prod_{k=j}^{n} a_k, \text{ and } \prod_{k=1}^{\infty} a_k \tag{0.0.2}$$

mean?

0.3 What is a *geometric series*? If S is the infinite geometric series having initial term a and common ratio r,

$$S := a + ar + ar^2 + ar^3 + \cdots, \tag{0.0.3}$$

when does S converge? When S does converge, what is the value of S?

[1]For example, is π a *rational number* or an *irrational number*? An *algebraic number* or a *transcendental number*? Is π *simply normal* in base-ten (or any other particular base, and is it *normal* or *absolutely normal* in general? Feel free to offer your own properties of π, too!

0.4 Let n be a nonnegative integer. What is *n factorial*? *Notation:* n factorial is denoted $n!$.

0.5 Let p be an integer. What does it mean for p to be a *prime number*?

0.6 What is a *radian*? What is the relationship between radians and degrees?

0.7 What is a *zero of a function*?

0.8 What are the *sine* and *cosine functions*? What are their zeroes when expressed in radians?

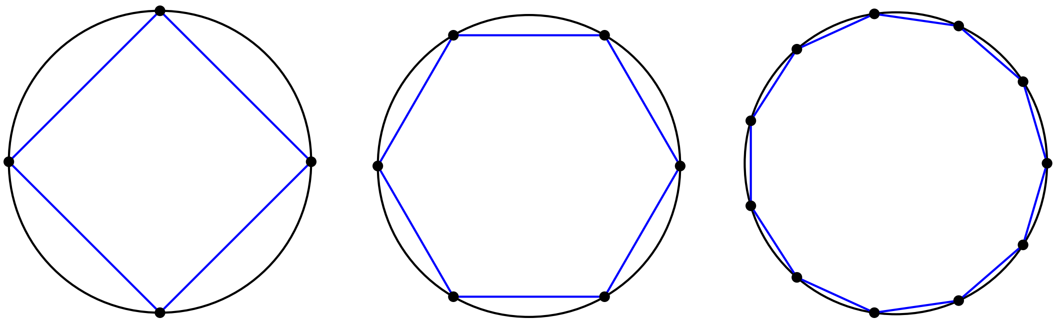

Figure 1.1.1: Inscribed 4, 6 and 11-gons in a circle of radius 1.

1 Limit of Polygons - Approximating the Circumference of a Circle

1.1 Polygons and Circles

One of the most natural ways for π to appear is in the circumference of a circle. For a circle of radius 1, we know that the circumference is 2π. Measuring the circumference of a circle is hard though. Instead, we are going to use a nice class of polygons to approximate the circle, and hopefully get a nice approximation of π as a result. First, we should define just what we are talking about.

Definition 1.1.1 (Regular Polygon)**.** We say that a polygon P, given by a collection of point vertices and straight lines joining them, is *regular* if it has all equal sides and all equal angles.

Definition 1.1.2 (Inscribed Circles)**.** We say that a polygon P is *inscribed* in a circle C if every vertex v in P is also in C.

Figure 1.1.1 shows three regular inscribed polygons in a circle of radius 1: a regular 4-gon (square), a 6-gon (hexagon), and a 11-gon. You can see that, as the number of sides increases, the polygon looks more and more like the circle it is inscribed in. By taking the perimeter of a many sided polygon, we can therefore approximate the circumference of a circle ... and therefore also approximate π!

1.1.1 What is the perimeter of a square inscribed in a circle of radius 1?

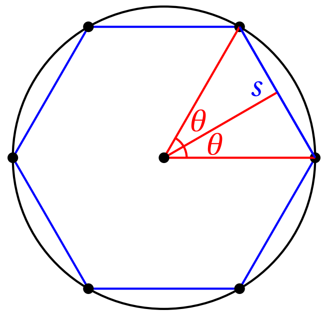

Figure 1.2.1: We can break up the n-gon into different right triangles.

1.1.2 How close is the perimeter of an inscribed square to the true circumference of 2π? You may need to know that $\sqrt{2} \approx 1.4$.

1.1.3 What is the side length of an regular hexagon inscribed in a circle of radius 1? Use this to make another approximation of π. Is this better or worse than what you got using a square?

1.2 Triangle Decomposition

In general, a good way to find the side length of a regular n-gon inscribed in a circle is to break up the shape into congruent right triangles, as shown in Figure 1.2.1. The interior angle of these triangle is labeled θ.

1.2.1 For an n-sided regular polygon broken up into these triangles, what is θ? For our purposes right now, we want to write θ in *radians*, not degrees. See the Exercise #0.6 for details if you have never seen radians before!

Figure 1.2.2 shows a schematic of a single one of these triangles. We can use trigonometry to write $\frac{s}{2}$ in terms of θ. Since we know the hypotenuse of the triangle and the side opposite to the angle, we know that SOH-CAH-TOA says

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\frac{s}{2}}{1} = \frac{s}{2}.$$

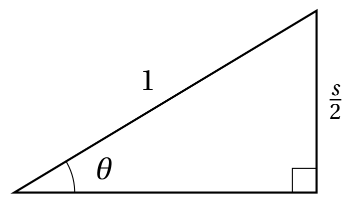

Figure 1.2.2: A single right triangle from the decomposition of our n-gon.

Using your result from the previous problem, we can then write

$$\text{Perimeter} = n \cdot s = 2n\sin(\theta) = 2n\sin\left(\frac{\pi}{n}\right).$$

As n gets really big (or equivalently as $1/n$ gets really small), we should expect this number to approach 2π. However, to calculate what this number is we *already have to know* π. Luckily for us, we have a sneaky way around this.

1.2.2 **Challenge for those who know some calculus:** Use the sum formula for sine given in Equation 3.2.4 and the change of variables $\pi/n = x$ to prove that

$$\lim_{n\to\infty} 2n\sin\left(\frac{\pi}{n}\right) = 2\pi. \tag{1.2.1}$$

1.3 Tricky Half-Angle Rules

While we do not know what $\sin(\pi/n)$ is for arbitrary n, we *do* know what it is for $n = 2$:

$$\sin\left(\frac{\pi}{2}\right) = 1. \tag{1.3.1}$$

We also know the sine half-angle formula, which says that for any θ with $0 \le \theta \le \pi$,

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}}, \tag{1.3.2}$$

and the Pythagorean identity for sine and cosine:

$$\sin^2\theta + \cos^2\theta = 1. \tag{1.3.3}$$

Together, these say that

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\sqrt{1-\sin^2\theta}}{2}}. \tag{1.3.4}$$

Admittedly, this is pretty bad, but by applying this formula iteratively, it allows us to explicitly compute what $\sin\frac{\pi}{2^n}$ is for any n.

Example 1.3.1. Let's compute this for $n = 4$, meaning we have to do the above iteration scheme once. Since $\sin\left(\frac{\pi}{2}\right) = 1$,

$$\sin\frac{\pi}{4} = \sqrt{\frac{1-\sqrt{1-\sin^2\frac{\pi}{2}}}{2}} = \sqrt{\frac{1-\sqrt{1-1}}{2}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}. \tag{1.3.5}$$

This means that the perimeter of a square is given by

$$2n\sin\left(\frac{\pi}{n}\right) = 8\frac{\sqrt{2}}{2} = 4\sqrt{2}, \tag{1.3.6}$$

as you likely found out much more directly in an earlier problem.

By applying this iterative scheme, we can explicitly compute the perimeter of any regular 2^n-gon inscribed in a unit circle, and use those perimeters as an approximation for π!.

1.3.1 Take this scheme one step further to find the perimeter of an inscribed regular octagon.

2 Buffon's Needle Problem

2.1 Setup

Suppose you have a thin needle of length l and you are standing on a floor made of infinitely long vertical lines equally spaced apart with distance D. Now, being clumsy and

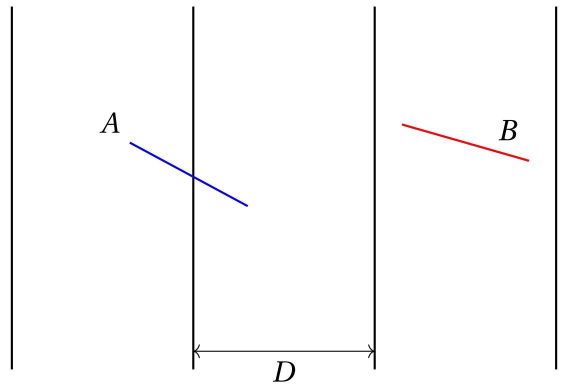

Figure 2.1.1: Two needles dropped on the floor (you're very clumsy). A has intersected the parallel lines, but B has not.

disoriented from standing in an infinitely long room, you drop the needle on the floor. What is the probability that the needle is intersecting one of the vertical lines on the floor?

This question was posed by the Count of Buffon in the 18th century, and it turns out that there is a nice answer related to, of all things, π! When $l \leq D$, the probability p of the needle intersecting a line is given exactly by

$$p = \frac{2l}{\pi D}. \tag{2.1.1}$$

You can actually approximate π by doing many trials of dropping needle-like objects on the floor and measuring the experimental probability of success! We'll explore now how exactly this formula is proven, although for the full proof we will need one fact from calculus.

2.1.1 What are the things you need to know about a needle in order to determine whether it intersects a line (position, angle, etc.)?

Suppose the needle gets dropped at a point (x, y) on the floor, and we measure the acute angle θ between it and the vertical direction. We notice that the y coordinate has no bearing on whether our needle intersects, so we may fully ignore it. Additionally, we only need to check when x is in the range $[0, \frac{D}{2}]$, since by symmetry the probability the needle intersects the line at 0 with x sampled from $[0, \frac{D}{2}]$ is the same as the original probability we desired. Even further, we need only check when $\theta \in [0, \frac{\pi}{2}]$ again by symmetry. Thus, we only need to examine $x \in [0, \frac{D}{2}]$ and $\theta \in [0, \frac{\pi}{2}]$.

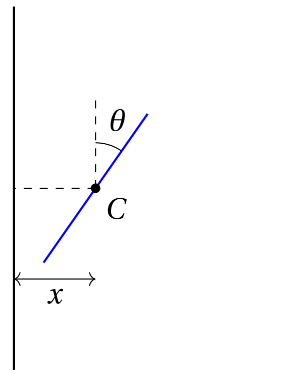

Figure 2.1.2: The Buffon Needle labeled with x and θ.

2.1.2 Use trigonometry to compute the horizontal distance h between the center and either endpoint of the needle.

By the above problem, we know that the needle intersects the vertical line at $x = 0$ if and only if $0 \leq x < h = \frac{l}{2}\sin\theta$. To find out how to use this to solve our problem, we're first going to look at some simpler examples.

2.2 Simpler Problems

Let's consider some simpler version of this problem. Suppose for that I have two random numbers x, θ both picked randomly and independently from $[0, 1]$, and we are interested in the probability that $\theta < x$.

There is a relatively nice way to do this that follows from noticing that the probability of $\theta < x$ should be equal by symmetry to the probability of $\theta > x$. Then, since the probability that $x = \theta$ exactly is 0, and since the sum of all possible probabilities is 1, we know that

$$1 = P(\theta < x) + P(\theta > x) + P(\theta = x) \tag{2.2.1}$$

$$= P(\theta < x) + P(\theta > x) + 0 \tag{2.2.2}$$

so $P(\theta < x) = \frac{1}{2}$.

However, we want to take a slightly different perspective. We can draw the space of all possible x and θ on a $2D$ graph, which in probability is often called the *sample space*, since it is where we sample our variables from. For each pair (x, θ), we can take note of whether $\theta < x$ (when our condition is TRUE) or $\theta \geq x$ (when our condition is FALSE). This will trace out two regions, one TRUE and one FALSE in the sample space.

It turns out that the relative area of the TRUE region *gives us exactly the probability that our statement is true.* Figure 2.2.1 shows what this looks like for the $\theta < x$ condition.

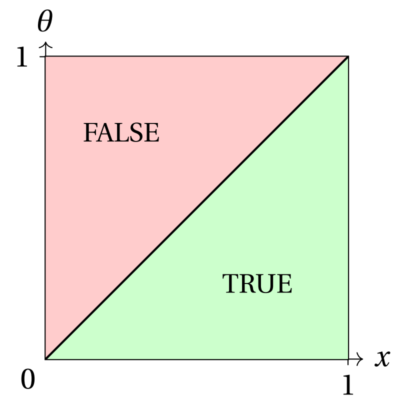

Figure 2.2.1: Sample space splitting into TRUE and FALSE regions.

We can then compute the probability $P(\theta < x)$ by noticing that the area of our sample space is the area of a 1×1 rectangle and that the area of our TRUE space is that of a 1×1 right triangle, so

$$\begin{aligned} P(\theta < x) &= \frac{\text{Area(TRUE)}}{\text{Area(TOTAL)}} \\ &= \frac{\frac{1}{2} \cdot 1 \cdot 1}{1 \cdot 1} \\ &= \frac{1}{2}, \end{aligned}$$

just as we computed earlier.

2.2.1 Use the sample space area technique to find the probability that $\theta < \frac{1}{2}$.

2.2.2 What if our condition were instead that $x^2 + \theta^2 < 1$?

Hint: The equation $x^2 + \theta^2 = 1$ traces out a circle of radius 1 in the x, θ plane centered at 0.

2.3 Buffon Revisited

Armed with our new tools, we can attempt to finish the Buffon problem.

2.3.1 Draw the sample space for the Buffon problem and label the TRUE and FALSE regions. Remember that now x and θ are not both from $[0,1]$ anymore!

This looks very unlike all of our other examples, but if you swap the position of x and θ, you get something much more normal, as shown in Figure 2.3.1.

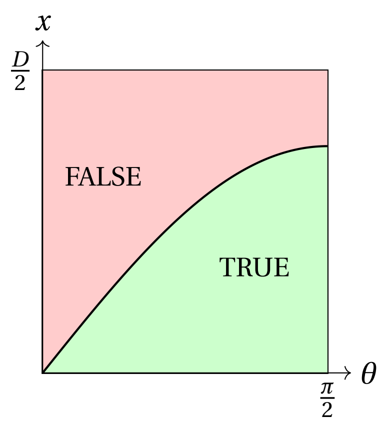

Figure 2.3.1: Sample space for the Buffon needle problem, after swapping x and θ.

The key ingredient of calculating the probability, then, is computing the area bounded between 0 and $\frac{l}{2}\sin\theta$ for $\theta \in [0, \frac{\pi}{2}]$. In calculus, the operation that gives you the area under a curve is called *integration*, and is represented by a big swoopy s kind of sign. So the area under the curve for some function $f(x)$ between $x = a$ and $x = b$ is usually written

$$\int_a^b f(x)\,dx, \tag{2.3.1}$$

where the dx helps let you know that you are *integrating* in the x direction[2].

Then it is a nice fact from calculus that

$$\int_0^{\frac{\pi}{2}} \sin(x)\,dx = 1. \tag{2.3.2}$$

[2]This is also to make the fundamental theorem of calculus look nicer, since integration throws in a dx to counter the derivative's $\frac{d}{dx}$.

From there, we can compute that

$$\text{Area(TRUE)} = \frac{l}{2}\int_0^{\frac{\pi}{2}} \sin(x)\,dx = \frac{l}{2}. \tag{2.3.3}$$

Then since the total area of our sample space is $\frac{\pi D}{4}$, we conclude that

$$P(x < \frac{l}{2}\sin(\theta)) = \frac{\text{Area(TRUE)}}{\text{Area(TOTAL)}} = \frac{\frac{l}{2}}{\frac{\pi D}{4}} = \frac{2l}{\pi D}. \tag{2.3.4}$$

This technique of using areas and integration (the act of taking an integral) to compute probabilities is *super* strong, and used all the time in more advanced probability theory, and can even be used to prove all of the results you know about from discrete probability too!

3 The Basel Problem

3.1 Discussion

The constant π is typically introduced via geometry and circles, so it can be surprising when it arises in contexts that seem to have no direct connection. In this section, we present one such example:

Problem (The Basel Problem[3]). Compute, with justification, the value of the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots. \tag{3.1.1}$$

Much of our discussion, as well as the method presented in the exercises, is due to the Swiss mathematician *Leonhard Euler*[4] *(1707–1783)*. To provide the foundation for this argument, we shall follow Euler's strategy as presented in Chapter 9 of *Journey through Genius: The Great Theorems of Mathematics* by William Dunham ([1]):[5]

- First, we need to connect the *roots* (or *zeroes*) of a *polynomial* to how to factor that polynomial.

[3]This problem was first posed in 1650 by Italian mathematician and Catholic priest *Pietro Mengoli (1626–1686)*. It is named for the Swiss city *Basel*, original home to Leonhard Euler (who first solved it) and *the Bernoulli family* (several of whose members taught Euler).

[4]"Euler" is pronounced "oiler". This has inspired mathematically themed wordplay regarding the NHL's Edmonton Oilers.

[5]A personal note from David: I strongly recommend *Journey through Genius* to students in the advanced group. It alternates between biographies of famous mathematicians, from *Hippocrates of Chios (c. 470–c. 421 BCE)* to *Georg Cantor (1845–1918)*, and their proofs of some of their most important results, from Euclid's proof there are infinitely many primes to Cardano's cubic formula to the Bernoullis' proof that the *harmonic series* diverges.

- Next, we need to introduce the *sine* function (whose argument is measured in radians), especially noting for which real x we have $\sin x = 0$.

- Next, we consider the function

$$f(x) := \begin{cases} 1, & \text{if } x = 0, \\ \dfrac{\sin x}{x}, & \text{otherwise.} \end{cases} \tag{3.1.2}$$

  We then express $f(x)$ as a *power series* (centered at $x = 0$).

- Finally, we introduce Euler's creative masterstroke: generalize from properties of factoring polynomials (which by definition have only finitely many nonzero terms) and factor the "infinite polynomial" that is our power series for $f(x)$. Via clever manipulation of the result, we shall be able to compute the sum in Equation (3.1.1).

Caveat. Euler's approach is ingenious, and it does correctly compute the sum in Equation (3.1.1). His manipulation of infinite series and products, however, is *not* rigorous[6] by modern standards.

We begin by presenting the following, taken from our session "Vieta's Formulas and Symmetric Polynomials" from March 15, 2025, without proof:

Theorem 3.1.1 (Polynomial Remainder Theorem)**.** *Let $p(x)$ be a polynomial with complex coefficients, and let $c \in \mathbb{C}$. Then the remainder when dividing $p(x)$ by $x - c$ is the constant $p(c)$.*

That is, p is expressible in the form $p(x) = (x - c)q(x) + p(c)$ for some polynomial q, and where $p(c)$ is the constant.

Note: Our priority is that you understand and can use this theorem later. Being able to prove it would be a bonus, but secondary.

Corollary 3.1.1(a). *If $p(x)$ is a polynomial with complex coefficients, and if $c \in \mathbb{C}$, then $x - c$ divides p if and only if $p(c) = 0$.*

[6]This should be unsurprising, since recognizably modern definitions of limit did not arise until years after Euler's death in 1783. The modern approach is typically credited to Bohemian mathematician *Bernard Bolzano (1781–1848)* in 1817, French mathematician, engineer, and physicist *Augustin-Louis Cauchy (1789–1857)* in 1821, and German mathematician *Karl Weierstrass (1815–1897)* in 1861.

Note: Again, the priority is being able to understand and apply this Corollary, not prove it.

3.2 Exercises

3.2.1 Let p be a polynomial of degree $n \geq 1$. Prove that if p has the n roots (including *multiplicity*[7]) $c_1, c_2, \ldots, c_n$, and if $p(0) = 1$, then we can factor $p(x)$ in the form

$$p(x) = \left(1 - \frac{x}{c_1}\right) \cdot \left(1 - \frac{x}{c_2}\right) \cdot \cdots \cdot \left(1 - \frac{x}{c_n}\right). \tag{3.2.1}$$

Remark. Note that since $p(0) = 1$, this implies that $c_j \neq 0$ for each root c_j of p. (After all, if $c_j = 0$ for some root c_j of p, then by the definition of a root of a polynomial, we must have $p(0) = 0$.) As a result, in the right-hand side of Equation (3.2.1), none of the factors $\left(1 - \frac{x}{c_j}\right)$ involves dividing by 0.

3.2.2 We present the following result without proof:

Proposition 3.2.1. *Let x be measured in radians. Then for all x,*

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots, \tag{3.2.2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots. \tag{3.2.3}$$

[7] I.e., taking repeated roots into account according to how often a root is repeated. For example, the polynomial $p(x) = (x-1)^2(x-2)^3$ has the two roots $x = 1$ and $x = 2$. The former has multiplicity 2, and the latter has multiplicy 3.

In sigma notation, *the above are expressible as*

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \tag{3.2.4}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}. \tag{3.2.5}$$

In particular, the sums in Proposition 3.2.1 *converge* for all real numbers x. (Indeed, the series converges at every complex number, too.)

Let $f(x)$ be defined as in Equation (3.1.2) above. Using Proposition 3.2.1, what is the power series for $f(x)$?

3.2.3 Consider the function $\sin x$, where x is expressed in *radians*. What are the zeroes of the sine function? That is, find all solutions x to the equation

$$\sin x = 0. \tag{3.2.6}$$

(Compare to Exercise #0.7 from the Warmup.) Using this, determine all the zeroes to the function $f(x)$ defined in Equation (3.1.2).

Remark. The sine function can be extended to a function on the set of all *complex numbers*, denoted $\mathbb{C}$, using the power series in Proposition 3.2.1. Whether or not you are familiar with the complex sine function, you may assume here that the only zeroes, real *or* complex, of the sine function are *real* numbers.

3.2.4 In the following, we seek to compute the coefficient of x^2 upon expanding the following polynomial products. For each example, can you compute this coefficient *without* fully multiplying out the product?

(a) Let a, b be constants. When expanding

$$(1 - ax^2) \cdot (1 - bx^2), \tag{3.2.7}$$

what is the resulting coefficient of x^2?

(b) Let a, b, c be constants. When expanding

$$(1 - ax^2) \cdot (1 - bx^2) \cdot (1 - cx^2), \tag{3.2.8}$$

what is the resulting coefficient of x^2?

(c) Let a, b, c be constants. When expanding

$$(1 - ax^2) \cdot (1 - bx^2) \cdot (1 - cx^2) \cdot (1 - dx^2), \tag{3.2.9}$$

what is the resulting coefficient of x^2?

(d) Let n be a positive integer, and $a_1, a_2, \ldots, a_n$ constants. When expanding

$$(1 - a_1 x^2) \cdot (1 - a_2 x^2) \cdot \cdots \cdot (1 - a_n x^2), \tag{3.2.10}$$

what is the resulting coefficient of x^2? Consider the pattern in your answers to Exercises #3.2.4(a)–3.2.4(c).

3.2.5 In Exercise #3.2.2, you should have found that for $f(x)$ as defined in Equation (3.1.2),

$$f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k}. \tag{3.2.11}$$

From Exercise #3.2.3, we determined that c is a zero of $f(x)$ if and only if $c = k\pi$, where k is any *nonzero* integer. Note also that, as in Exercise #3.2.1, $f(0) = 1$ (though f is *not* a polynomial).

(a) Generalizing from your answer to Exercise #3.2.4(d), use these results to explain why

$$f(x) = \left(1 - \frac{x}{\pi}\right) \cdot \left(1 - \frac{x}{-\pi}\right) \cdot \left(1 - \frac{x}{2\pi}\right) \cdot \left(1 - \frac{x}{\pi}\right) \cdot \cdots \cdot \left(1 - \frac{x}{k\pi}\right) \cdot \left(1 - \frac{x}{-k\pi}\right) \cdot \cdots. \tag{3.2.12}$$

(b) For every positive integer n, show that

$$\left(1 - \frac{x}{k\pi}\right) \cdot \left(1 - \frac{x}{-k\pi}\right) = 1 - \frac{x^2}{k^2\pi^2}. \tag{3.2.13}$$

Conclude that

$$f(x) = \left(1 - \frac{1}{1^2\pi^2}\right) \cdot \left(1 - \frac{1}{2^2\pi^2}\right) \cdot \cdots \cdot \left(1 - \frac{1}{k^2\pi^2}\right) = \prod_{k=1}^{\infty} \left(1 - \frac{1}{k^2\pi^2}\right) \tag{3.2.14}$$

(c) Assume that the pattern for a product of finitely many terms from Exercise #3.2.4(d) continues for infinite products of the analogous form. In expanding the infinite product in Equation (3.2.14), what is the coefficient of x^2?

3.2.6 Comparing our results from Exercises #3.2.2 and #3.2.5(c), equate the coefficients of x^2. From this, complete Euler's solution to the Basel Problem:

Theorem 3.2.2 (Euler, published 1735)**.**

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}. \tag{3.2.15}$$

3.2.7 Having computed the sum in Equation (4.1.1), we can now compute two related infinite series: the sum of the reciprocals of the *even* positive integers, and the sum of the reciprocals of the *odd* positive integers, respectively.

(a) Compute the sum

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \cdots. \tag{3.2.16}$$

Hint: Note that

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \sum_{k=1}^{\infty} \left(\frac{1}{2^2} \cdot \frac{1}{k^2} \right) = \frac{1}{4} \cdot \sum_{k=1}^{\infty} \frac{1}{k^2}. \tag{3.2.17}$$

Based on what we have already proven, what can you conclude?

(b) Compute the sum

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots. \tag{3.2.18}$$

3.2.8 Euler's method for solving the Basel Problem can also be used to compute *the Wallis Product*[8]:

$$\left(\frac{2}{1}\cdot\frac{2}{3}\right)\cdot\left(\frac{4}{3}\cdot\frac{4}{5}\right)\cdot\left(\frac{6}{5}\cdot\frac{6}{7}\right)\cdot\left(\frac{8}{7}\cdot\frac{8}{9}\right)\cdots=\prod_{k=1}^{\infty}\frac{4k^2}{4k^2-1}=\prod_{k=1}^{\infty}\left(\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right). \quad (3.2.19)$$

Consider the infinite product representation in Equation (3.2.14). Recall, too, from our definition of $f(x)$ that that for $x \neq 0$, $f(x) = \frac{\sin x}{x}$. Evaluate

$$f\left(\frac{\pi}{2}\right) \quad (3.2.20)$$

in two different ways, and conclude that the Wallis Product has value

$$\left(\frac{2}{1}\cdot\frac{2}{3}\right)\cdot\left(\frac{4}{3}\cdot\frac{4}{5}\right)\cdot\left(\frac{6}{5}\cdot\frac{6}{7}\right)\cdot\left(\frac{8}{7}\cdot\frac{8}{9}\right)\cdots=\frac{\pi}{2}. \quad (3.2.21)$$

4 The Riemann Zeta Function and π in Number Theory

4.1 Discussion

In Section 3, we saw that π arises in a potentially surprising way when computing an infinite sum:

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}. \quad (4.1.1)$$

In this section, we extend on our techniques from Section 3 to compute other sums. We then use these ideas to make some computations of probabilities arising from number theory.

First, we generalize the sum from the Basel Problem, Equation (3.1.1), replacing the exponent 2 to an arbitrary exponent $s > 1$:

Definition 4.1.1. Let s be a real number with $s > 1$. The *Riemann*[9] *zeta function* evaluated at s, denoted $\zeta(s)$, is defined by the sum

$$\zeta(s) := \sum_{k=1}^{\infty}\frac{1}{k^s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots. \quad (4.1.2)$$

[8]Named for *John Wallis (1616–1703)*, an English clergyman and mathematician.

[9]Named for *Bernhard Riemann (1826–1866)* (pronounced "REE-mahn ZAY-tuh"), a German mathematician who made enormous contributions to several branches of mathematics.

That is, the Riemann zeta function at $s > 1$ is the sum of the reciprocals of all the s-powers of the positive integers.

Remark. In particular, the infinite sum above converges for all $s > 1$. Note that as s increases, the denominators in the sum in Equation (4.1.2) also get bigger, meaning the overall sum gets smaller; that is, ζ is a strictly decreasing function on the interval $s \in (1, +\infty)$.

There is a further generalization of the Riemann zeta function for values of s that are complex numbers. Limiting our consideration to real values of s with $s > 1$ suffices for our purposes here.

Example 4.1.2. By Definition 4.1.1, we can interpret the Basel Problem as asking us to compute the value $\zeta(2)$. Equation (4.1.1) above can be rewritten in the form

$$\zeta(2) = \frac{\pi^2}{6}. \tag{4.1.3}$$

Section 3 led us to a method for computing $\zeta(2)$. In this section, we seek to compute $\zeta(n)$ for additional values where n is a positive integer. (As in Section 3, we shall follow the approach in [1].) This shall culminate in providing an answer to the following problem from number theory—which, remarkably, is relevant to the Riemann zeta function!

4.2 Exercises

4.2.1 To begin, let us generalize our results from Exercise #3.2.4 to help us compute

$$\zeta(4) = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots. \tag{4.2.1}$$

To compute $\zeta(2)$, we were interested in the coefficient of x^2 in the power series for $f(x)$. Motivated by that, to compute $\zeta(4)$, let us consider the coefficient of x^4 in $f(x)$.

(a) Let a, b be constants. When expanding $(1 - ax^2) \cdot (1 - bx^2)$ (also seen above in Equation (3.2.7)), what is the coefficient of x^4?

(b) Continuing Exercise #4.2.1(a), prove that the coefficient of x^4 in $(1 - ax^2) \cdot (1 - bx^2)$ is

$$\frac{1}{2}\left[(a+b)^2 - (a^2 + b^2)\right]. \tag{4.2.2}$$

(c) Let a, b, c be constants. When expanding $(1-ax^2)\cdot(1-bx^2)\cdot(1-cx^2)$, what is the resulting coefficient of x^4?

(d) Continuing Exercise #4.2.1(c), prove that the coefficient of x^4 in $(1-ax^2)\cdot(1-bx^2)$ is

$$\frac{1}{2}\cdot\left[(a+b+c)^2-(a^2+b^2+c^2)\right]. \tag{4.2.3}$$

(e) Let a, b, c, d be constants. When expanding $(1-ax^2)\cdot(1-bx^2)\cdot(1-cx^2)\cdot(1-dx^2)$, what is the resulting coefficient of x^4?

(f) Let n be a positive integer and $a_1, a_2, \dots, a_n$ constants. When expanding $(1-a_1x^2)\cdot(1-a_2x^2)\cdot\dots\cdot(1-a_nx^2)$, what is the resulting coefficient of x^4?

(g) Continuing Exercise #4.2.1(f), prove that the coefficient of x^4 in $(1-a_1x^2)\cdot(1-a_2x^2)\cdot\dots\cdot(1-a_nx^2)$ is

$$\frac{1}{2}\left[(a_1+a_2+\cdots+a_n)^2-(a_1^2+a_2^2+\cdots+a_n^2)\right]. \tag{4.2.4}$$

(h) Modifying our computations in solving Exercise #3.2.6, prove that

$$\zeta(4) = \frac{\pi^4}{90}. \tag{4.2.5}$$

(i) **Challenging:** Using these techniques, can you compute $\zeta(6)$? How about $\zeta(2n)$ for larger even integers $2n$?

Remark. Using this technique, Euler computed the value of $\zeta(2n)$ for even positive integers $2n \le 26$. In general, for every even positive integer of this form, $\zeta(2n)$ is of the form

$$\zeta(2n) = \frac{a}{b} \cdot \pi^{2n}, \tag{4.2.6}$$

where a, b are positive values. That is, for each even positive integer of the form $2n$, $\zeta(2n)$ is always a *rational* multiple of π^{2n}. In particular, since π is not merely irrational but transcendental (as a consequence of the *Lindemann–Weierstrass Theorem*[10]), $\zeta(2n)$ is a transcendental (and thus irrational) number for every positive even integer $2n$.

During the centuries since Euler's proof, comparatively limited progress has been made in understanding the value of the Riemann zeta function at odd positive integers. Euler was unable to use his method to compute a closed-form expression for $\zeta(3)$, nor for $\zeta(2n-1)$ for any positive odd integer of the form $2n-1$ with $n \ge 2$.

[10] This theorem, whose statement is beyond the scope of this session, is named for German mathematicians *Ferdinand von Lindemann (1852–1939)* and *Karl Weierstrass (1815–1897)*. The theorem is also sometimes called the *Hermite–Lindemann Theorem* or the *Hermite–Lindemann–Weierstrass Theorem*, recognizing French mathematician *Charles Hermite (1822–1901)*, who proved a simpler case of the general theorem.

Note: German pronunciation rules mean that "Weierstrass" is pronounced like VIE-air-shtrahss, though anglicized pronunciations like WIE-er-strahss are common. Further, you may encounter his name spelled as "Weierstraß", especially in German texts; the *eszett/double s/sharp s* character *ß* is roughly equivalent to "ss". (This also applies to other Germans, notably *Carl Friedrich Gauss (1777–1855)*, whose name often appears in German-language texts as "Gauß".)

Apéry's[11] *Theorem (1978)* established that $\zeta(3)$ is irrational, so $\zeta(3)$ is also called *Apéry's constant*. French mathematician *Tanguy Rivoal (1972–)* proved in 2000 that infinitely many values amongst $\zeta(3), \zeta(5), \zeta(7), \zeta(9), \zeta(11), \ldots$ are irrational, and in 2001, Russian mathematician *Wadim Zudilin* proved that at least one of the four values $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, and $\zeta(11)$ is irrational.

4.2.2 Our goal in this exercise is to build a solution to the following problem:

Problem 4.2.1. Let a, b be randomly chosen[12] among the set of natural numbers, $\mathbb{N} := \{1, 2, 3, \ldots\}$. What is the probability that a and b are *relatively prime*?

More generally, if $n > 1$ is a positive integer and $a_1, a_2, \ldots, a_n$ are randomly selected positive integers, what is the probability that $\gcd(a_1, a_2, \ldots, a_n) = 1$?

Warning. Note that we may have that $\gcd(a_1, a_2, \ldots, a_n) = 1$ even if $\gcd(a_i, a_j) > 1$ for each pair of distinct indices i, j. For example, $\gcd(6, 10, 15) = 1$, since there is no nontrivial common divisor to 6, 10, and 15 simultaneously. However, $\gcd(6, 10) = 2 > 1$, $\gcd(6, 15) = 3 > 1$, and $\gcd(10, 15) = 5 > 1$.

A collection of integers $\{a_1, a_2, \ldots, a_n\}$ such that $\gcd(a_1, a_2, \ldots, a_n) = 1$ is called *relatively prime*. If we also have that $\gcd(a_i, a_j) = 1$ whenever $i \neq j$, then this stronger condition is that the a_j are *pairwise relatively prime*. Our assertion in Problem 4.2.1 concerns the weaker condition that $\{a_1, a_2, \ldots, a_n\}$ is relatively prime but not necessarily pairwise relatively prime.

To elaborate on what "randomly chosen" in Problem 4.2.1 means, we take the following approach. It will use the notion of the *limit of a sequence* from calculus, but in a way that ought to be intuitively understandable, even if not fully understood with rigorous technical detail.

Definition 4.2.2. Fix N as a (typically large) positive integer, and consider $S_N := \{1, 2, \ldots, N\}$. For fixed N, we select a, b uniformly at random.[13] For fixed N, consider the probability

$$P_N := P(\gcd(a, b) = 1 : a, b \in S_N). \tag{4.2.7}$$

That is, for a fixed positive integer N, P_N denotes the probability that if a, b are randomly chosen from within the finite set $\{1, 2, \ldots, N\}$, then P_N denotes the probability that $\gcd(a, b) = 1$.

[11] *Roger Apéry (1916–1994)* was a Greek-French mathematician.

[12] Establishing an "objective" sense of what "randomly chosen" means in an infinite set is more subtle than one might first appreciate. For an example suggesting at least three natural methods for random selection, consider *Bertrand's Paradox*: if a chord is chosen at random from within a given circle, what is the probability its length will be longer than that of the side of an equilateral triangle inscribed within the circle? For more details about Bertrand's Paradox, see videos by Numberphile and Numberphile2 videos with Grant Sanderson of 3Blue1Brown here and here.

[13] I.e., in the sense of a *discrete random distribution*: every choice for both a and b shall be equally likely, and these choices shall be *independent* of the other. In particular, this means each *pair* (a, b) occurs with equal probability $\frac{1}{N} \cdot \frac{1}{N} = \frac{1}{N^2}$.

Let

$$P(\gcd(a,b)=1 : a,b \in \mathbb{N}) \tag{4.2.8}$$

denote *the probability that a, b are relatively prime if chosen randomly among* all *positive integers.* Then we define this probability to be the limit

$$P(\gcd(a,b)=1 : a,b \in \mathbb{N}) := \lim_{N\to\infty} P_N, \tag{4.2.9}$$

provided the limit exists.

We may often let P denote the probability in Equation (4.2.8) to simplify our notation.

In Section 3, we expressed the function $f(x)$ from Equation (3.1.2) as both a power series (in Exercise #3.2.2 and Equation (3.2.11)), as well as an infinite product (Equation (3.2.13) in Exercise #3.2.5(b)). We now wish to express, for $s > 1$, $\zeta(s)$ as an infinite product. This time, though, our infinite product will indexed over the positive prime numbers, not all positive integers. Let us proceed, step by step, with a solution to Problem 4.2.1.

(a) Let p be a positive prime. In the context of Exercise #0.3 from the Warmup, consider the infinite geometric series

$$\sum_{k=0}^{\infty} \frac{1}{p^k} = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots. \tag{4.2.10}$$

Does this sum converge? If so, what is the value of this sum?

(b) Again, let p be a positive prime, and let $s > 1$. Generalize your result from Exercise #4.2.2(a) to show that

$$\sum_{k=0}^{\infty} \frac{1}{p^{ks}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots$$

$$= \frac{1}{1 - \frac{1}{p^s}} \tag{4.2.11}$$

$$= \frac{p^s}{p^s - 1}. \tag{4.2.12}$$

(c) Let $s > 1$, and let $p_1 := 2$, $p_2 := 3$, $p_3 := 5$, with, in general, $p_k :=$ the kth positive prime in ascending order. Consider the product

$$\left(\sum_{j=0}^{\infty} \frac{1}{p_1^{js}}\right) \cdot \left(\sum_{k=0}^{\infty} \frac{1}{p_2^{ks}}\right) = \left(1 + \frac{1}{p_1^s} + \frac{1}{p_1^{2s}} + \frac{1}{p_1^{3s}} + \cdots\right) \cdot \left(1 + \frac{1}{p_2^s} + \frac{1}{p_2^{2s}} + \frac{1}{p_2^{3s}} + \cdots\right). \tag{4.2.13}$$

Explain that when expanding the product of infinite series in Equation (4.2.13), every term is of the form

$$\frac{1}{(p_1^j p_2^k)^s}, \tag{4.2.14}$$

where j, k are nonnegative integers. Further, every term of the form in Equation (4.2.14) appears *precisely* once when expanding the product of these infinite series.

(d) Following the notation in Exercise #4.2.2(c), explain how when expanding the product

$$\left(\sum_{j=0}^{\infty} \frac{1}{p_1^{js}}\right) \cdot \left(\sum_{k=0}^{\infty} \frac{1}{p_2^{ks}}\right) \cdot \left(\sum_{\ell=0}^{\infty} \frac{1}{p_3^{\ell s}}\right), \tag{4.2.15}$$

the general term in this expansion is of the form

$$\frac{1}{(p_1^j p_2^k p_3^\ell)^s}, \tag{4.2.16}$$

where j, k, ℓ are nonnegative integers. Further, every term of the form in Equation (4.2.16) appears *precisely once* when expanding the product of these three infinite series.

(e) *Assuming* that we may continue this process to the infinite product *of* infinite series, explain how the general term is of the form

$$\frac{1}{n^s}, \tag{4.2.17}$$

where n is *any* positive integer. Further, every term of the form in Equation 4.2.17 appears *precisely once* in that the infinite product of infinite series.

(f) From Exercise #4.2.2(e), deduce the *Euler product* for the Riemann zeta function: for all $s > 1$,

$$\zeta(s) = \prod_{\substack{p>0 \\ p \text{ prime}}} \frac{1}{1 - \frac{1}{p^s}} \tag{4.2.18}$$

That is, the Riemann zeta function, defined in Definition 4.1.1 as an infinite series, is expressible as an infinite product, where are indices in the product are precisely the positive prime numbers.

4.2.3 Using the Euler product for the Riemann zeta function, Equation (4.2.18), we will now be able to compute certain probabilities regarding number theoretic properties.

(a) Let a, b be positive integers. Prove that $\gcd(a, b) = 1$ if and only if there is *no* positive prime p such that p is a divisor of a and b simultaneously.

Equivalently, prove that $\gcd(a, b) > 1$ if and only if there exists some positive prime p such that p is a divisor of both a and b simultaneously.

(b) Let p be a positive prime, and select a positive integer N. Show that if a is chosen randomly from the set $S_N = \{1, 2, \ldots, N\}$, then the probabilty that a is divisible by p is

$$P(p \text{ divides } a : 1 \leq a \leq N) = \frac{\left\lfloor \frac{N}{p} \right\rfloor}{N}, \tag{4.2.19}$$

Where $\left\lfloor \frac{N}{p} \right\rfloor$ is the *floor* or *greatest integer* of $\frac{N}{p}$. From this, explain why

$$P(p \text{ divides } a : a \in \mathbb{N}) = \frac{1}{p}. \tag{4.2.20}$$

(c) Fix a positive prime p, and let a, b be selected randomly from $\mathbb{N}$. Show that

$$P(a, b \text{ are } \textit{both} \text{ divisible by } p : a, b, \in \mathbb{N}) = \frac{1}{p^2}. \tag{4.2.21}$$

Conclude that

$$P(a, b \text{ are not } \textit{both} \text{ divisible by } p : a, b, \in \mathbb{N}) = 1 - \frac{1}{p^2}. \tag{4.2.22}$$

(d) Fix distinct positive primes $p_1, p_2, \ldots, p_k$, and let a, b be randomly selected from $\mathbb{N}$. Generalize from our result in Exercise #4.2.3(c) to show that

$$P(a, b \text{ are not } \textit{both} \text{ divisible by any of } p_1, p_2, \ldots p_k : a, b \in \mathbb{N})$$
$$= \left(1 - \frac{1}{p_1^2}\right) \cdot \left(1 - \frac{1}{p_2^2}\right) \cdot \cdots \cdot \left(1 - \frac{1}{p_k^2}\right). \tag{4.2.23}$$

Using Exercise #4.2.3(a), conclude that

$$P(\gcd(a, b) = 1 : a, b \in \mathbb{N}) = \prod_{\substack{p>0 \\ p \text{ prime}}} \left(1 - \frac{1}{p^2}\right). \tag{4.2.24}$$

(e) Combine Equation (4.2.24) with the formula for the Euler product in Equation (4.2.18) to explain why

$$P(\gcd(a,b)=1 : a,b\in\mathbb{N}) = \frac{1}{\zeta(2)}. \tag{4.2.25}$$

Conclude from our solution to the Basel Problem and Example 4.1.2 that

$$P(\gcd(a,b)=1 : a,b\in\mathbb{N}) = \frac{6}{\pi^2} \approx 0.6079. \tag{4.2.26}$$

That is, if a, b are randomly chosen positive integers (in the sense explained above), then the probability that a, b are relatively prime is about 60.79%.

(f) **Challenging:** Let $n \geq 2$ be a positive integer. Generalize the above argument to show that if $a_1, a_2, \ldots, a_n$ are randomly chosen from $\mathbb{N}$, then

$$P(\gcd(a_1, a_2, \ldots, a_n)=1 : a_1, a_2, \ldots a_n \in\mathbb{N}) = \frac{1}{\zeta(n)}. \tag{4.2.27}$$

Much more can be said about the Riemann zeta function. In particular, *The Riemann Hypothesis*, a conjecture about the zeroes over the complex number of the zeta function, is an open problem in mathematics. If true, it has a number of theoretical consequences, including about the distribution of the prime numbers. The Riemann Hypothesis is one of the six *Millennium Prize Problems* that are still unsolved, and a solution to any of these would earn a $1 million prize from the *Clay Mathematics Institute*.

4.2.4 We finally use Equation (4.2.26) to provide a method for estimating π.

(a) Reverse-engineering Equation (4.2.26), show that

$$\pi = \sqrt{\frac{6}{P(\gcd(a,b)=1 : a,b\in\mathbb{N})}}. \tag{4.2.28}$$

Since

$$P(\gcd(a,b)=1 : a,b\in\mathbb{N}) = \lim_{N\to\infty} P(\gcd(a,b)=1 : a,b\in S_N), \qquad (4.2.29)$$

deduce that

$$\pi = \lim_{N\to\infty}\sqrt{\frac{6}{P(\gcd(a,b)=1 : a,b\in S_N)}}. \qquad (4.2.30)$$

(b) To a pair of randomly chosen integers $a,b\in S_N$, associate it with the *lattice point*[14] (a,b) in the plane. Explain why $\gcd(a,b)=1$ if and only if the line segment from (a,b) to the origin $(0,0)$ does not intersect any any other lattice point between these endpoints.

(c) Exercises #4.2.4(a)–4.2.4(b) suggest a method for approximating π:

- First, take the $N\times N$ grid of lattice points of the form (a,b), where $1\leq a,b,\leq N$.
- Next, compute how many of these points represent relatively prime pairs by seeing which line segments from $(0,0)$ to (a,b) intersect some other lattice point; see Figure 4.2.1 to visualize this. The ratio of "relatively prime points" in this grid to N^2 will be approximate our probability $P(\gcd(a,b)=1 : a,b\in\mathbb{N}) = \frac{6}{\pi^2}$.
- Since

$$P(\gcd(a,b)=1 : a,b\in\mathbb{N}) \approx P(\gcd(a,b)=1 : a,b\in S_N) \qquad (4.2.31)$$

for sufficiently large N, we can approximate π by taking the ratio of the number of the "red points" (as in Figure 4.2.1) to N^2, the number of all "black points".

Choosing some small values of N, Table 4.2.1 gives the associated estimates to approximate π.

Can you use complete additional rows here?

[14] I.e., a point in the plane such that both its x-coordinate and y-coordinate are integers.

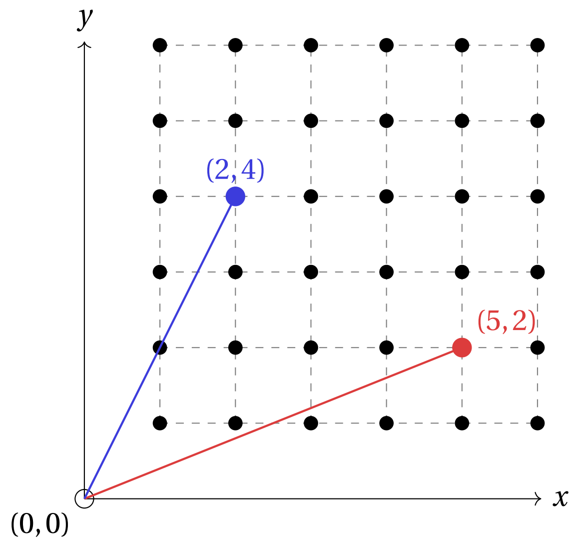

Figure 4.2.1: For $N = 6$: The red segment intersects no other black lattice points because $\gcd(5,2) = 1$. The blue line segment intersects another black lattice point because $\gcd(2,4) > 1$.

Remark. Those of you who attended our previous session, "Farey Fractions" on February 28, 2026, may recall that we had a formula to count the number of point in $\mathcal{F}_n$, the Farey sequence of order n. How might this help our computations above?

References

[1] William Dunham. *Journey through Genius: The Great Theorems of Mathematics*. John Wiley and Sons, Inc., 1990.

[2] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press, Walton Street, Oxford OX2 6DP, fifth edition, 1979.

[3] Ivan Niven, Herbert S. Zuckerman, and Hugh L. Montgomery. *An Introduction to the Theory of Numbers*. John Wiley & Sons, Inc., New York, fifth edition, 1991.

N	# coprime	total # points	$P(\gcd(a,b)=1 : a,b,\in S_N)$	est. π from (4.2.29)
1	1	1	1.0000	2.4495
2	3	4	0.7500	2.8284
3	7	9	0.7778	2.7775
4	11	16	0.6875	2.9542
5	19	25	0.7600	2.8010
6	23	36	0.6389	3.0645
7		49		
8		64		
9		81		
10		100		
11		121		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$N \to \infty$			$\frac{\pi^2}{6} \approx 0.6079$	$\pi \approx 3.1416$

Table 4.2.1: Estimating π using number theory and probability. (Values in the two right-most columns are approximations.)