Chapel Hill Math Circle Exit Ticket:
February 28, 2026: Farey Sequences

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- Was the this topic appropriately challenging relative to your background? That is, was the topic neither too elementary nor inaccessibly advanced?
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- What did you enjoy about today's topic?
- What did you find particularly challenging?
- Was there anything you thought was too difficult?
- Was there anything you thought was too easy?
- Are there any topics you would be interested in seeing us cover in the future?

Farey Sequences

Abstract

In this session, we shall introduce the *Farey sequence of order n*, denoted $\mathcal{F}_n$, and explore its properties. We shall introduce the concepts of *mediant* and *Ford Circles*, while recalling *Pick's Theorem*, a past topic at one of our previous sessions.

Background needed: Prerequisites include basic algebra and, ideally, a bit of number theory. The geometric concepts include that of the *slope* of a line, as well as *tangent circles* to circles and *tangent lines* to circles. In particular, it will be useful to know the equation for a circle in the xy-plane with center (a, b) and radius $r > 0$, as well as how to compute the distance between points (a, b) and (c, d) in the plane.

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Contents

0 Warmup

As prerequisites for this session, it will help to answer the following first. You do *not* need to know these answers already, and many questions will be revisited later in the worksheet.

0.1 What is the *slope* of a (nonvertical) line in the plane?

0.2 Let $P := (a, b)$, $Q := (c, d)$ be two points in the plane. What is the formula for the usual distance between P and Q?

0.3 Let (a, b) be a point in the plane, and $r > 0$. What is an equation for the unique circle with center (a, b) and radius r?

0.4 What is *mathematical induction*?

1 An Introduction to Farey Sequences

1.1 Discussion

Definition 1.1.1. Let n be a positive integer. The *Farey*[1] *sequence*[2] *of order n*, denoted $\mathcal{F}_n$, is the set of all lowest-terms quotients of the form $\frac{a}{b}$, where a, b are integers with $0 \le a \le b \le n$, and these quotients are arranged in increasing order. The sequence of all Farey sequences of order n, $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \dots)$, shall be called *the Farey sequence*.

That is, in the Farey sequence of order n, we arrange in increasing order all nonnegative lowest-terms fractions of the form $\frac{a}{b}$, $0 \le \frac{a}{b} \le 1$, where the denominator is at most n. *The* Farey sequence is the collection of all such sequences of order n, in ascending order of n.

Remark. Sources often choose definitions that appear to be different but which are ultimately equivalent. Compare, for example our definition, basically that of [3] to a more mediant-centric definition in [6]. (Namely, in forming $\mathcal{F}_{n+1}$ from $\mathcal{F}_n$, Hardy and Wright have you to insert the mediants between any two consecutive terms in $\mathcal{F}_n$, then discard all those whose lowest-terms denominators exceed n.)

Example 1.1.2. We have

$$\mathcal{F}_1 := \frac{0}{1}, \frac{1}{1} \tag{1.1.1}$$

$$\mathcal{F}_2 := \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \tag{1.1.2}$$

$$\mathcal{F}_3 = \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \tag{1.1.3}$$

Viewing these points as lying in the interval $[0, 1]$, we have the following:

The following function is relevant to the structure of Farey sequences:

Definition 1.1.3. Let $\varphi\colon \mathbb{N} \to \mathbb{N}$ be the function defined by

$$\varphi(n) := |\{k \in \mathbb{N} : 1 \le k \le n \text{ and } \gcd(k, n) = 1\}|. \tag{1.1.4}$$

[1] Named for John Farey Sr. (1766–1826), a British geologist. In 1816, Farey submitted for publication a conjecture about properties of his namesake sequence, what we present here as something equivalent to Theorem 2.2.1. Farey did not prove his conjecture, though, nor was he the first to study the sequence. Naming the sequence for Farey, then, is widely considered an example of *Stigler's Law of Eponymy*: "No scientific discovery is named after its original discoverer." (Fittingly enough, Stigler's law was itself discovered before Stigler, meaning the law is self-descriptive!)

[2] This has also often been called the *Farey series of order n*, though less commonly today. In mathematics, a *sequence* typically means an ordered list of items, where repetition is allowed. By contrast, a *series* typically connotes taking or computing a sum. We may be interested in taking the sum over the entries of a Farey sequence of order n—as in Exercise #1.2.7 below—but the sequence itself is the fundamental object of study, not the sum.

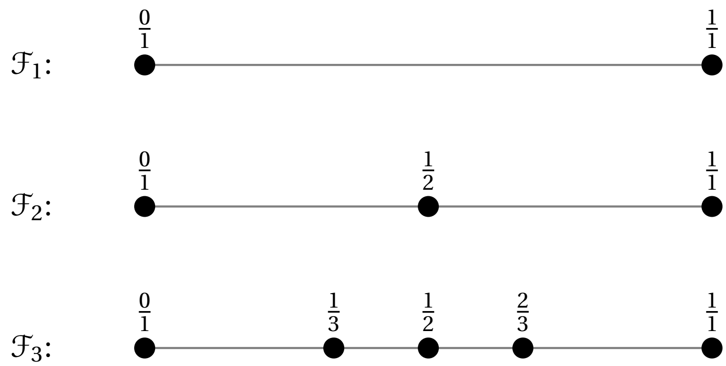

Figure 1.1.1: Visualizing $\mathcal{F}_n$ for small values of n.

That is, $\varphi(n)$ counts the number of positive integers in the set $\{1, 2, \ldots, n\}$ which are relatively prime to n. We call φ the *phi function* or *totient function*.

In particular, $\varphi(n)$ counts the number of distinct fractions among $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}$ which are already in lowest terms.

Example 1.1.4. For $n := 6$, the elements of $\{1, 2, 3, 4, 5, 6\}$ that are relatively prime to 6 are 1 and 5. Therefore, $\varphi(6) = 2$. Equivalently, consider the fractions $\frac{1}{6}, \frac{2}{6} = \frac{1}{3}, \frac{3}{6} = \frac{1}{2}, \frac{4}{6} = \frac{2}{3} = \frac{5}{6}, \frac{6}{6} = \frac{1}{1}$. Since only the two elements $\frac{1}{6}$ and $\frac{5}{6}$ are already in lowest terms, $\varphi(6) = 2$.

1.2 Exercises

1.2.1 What is $\mathcal{F}_4$? Which terms are are newly introduced to $\mathcal{F}_4$ compared to $\mathcal{F}_3$? How many new terms are added to $\mathcal{F}_3$ to form $\mathcal{F}_4$?

1.2.2 Repeat Exercise #1.2.1 for $\mathcal{F}_5$ and $\mathcal{F}_6$, respectively.

1.2.3 Complete Table 1.2.1, computing values of $\varphi(n)$ where φ denotes the function from Definition 1.1.3.

n	k in $\{1,2,\ldots,n\}$ satisfying $\gcd(k,n)=1$?	$\varphi(n)$?
1	1	1
2	1	1
3	1, 2	2
4	1, 3	2
5	1, 2, 3, 4	4
6	1, 5	2
7		
8		
9		
10		
11		
12		
13		
14		
15		
16		
17		
18		
19		
20		
⋮	⋮	⋮

Table 1.2.1: Computing $\varphi(n)$ for small values of n.

What patterns do you notice? Do you have any conjectures? Can you prove them?

1.2.4 Compare the new terms of F_{n+1} to the adjacent terms from F_n. What do you notice? Can you form a conjecture?

1.2.5 Let $\frac{a}{b}$, $\frac{c}{d}$ be consecutive terms in $\mathcal{F}_n$. What do you notice about these consecutive terms? Can you form a general conjecture?

1.2.6 If n is a positive integer, what is $|\mathcal{F}_n|$, the number of elements in the Farey sequence of order n? Express your answer in terms of the phi function from Definition 1.1.3 and Table 1.2.1.

1.2.7 Let n be a positive integer. What is the value of

$$\sum_{q \in \mathcal{F}_n} q, \tag{1.2.1}$$

the sum over all elements in $\mathcal{F}_n$?

2 Algebraic Tools

2.1 Discussion

We begin by presenting the following result without proof. Since the entire Farey sequence is defined recursively, with $\mathcal{F}_{n+1}$ defined in terms of which new terms are inserted from what we inherit from $\mathcal{F}_n$, it makes sense to think that mathematical induction may become a relevant proof method:

Theorem 2.1.1 (Mathematical Induction)**.** *Let* $\{P(n)\}$ *be a collection of statements for every positive integer* n*. Then* $P(n)$ *is true for every positive integer* n *if and only if*

(a) *$P(1)$ is true, and*

(b) *for every positive integer,* if *$P(n)$ is true,* then *$P(n+1)$ is true.*

Remark. Mathematical induction appeared in a supplement to our session from October 30, 2021, whose primary topic was *continued fractions*.

Definition 2.1.2. Let a, b, c, d be integers, with c, d positive integers, and the fractions $\frac{a}{b}, \frac{c}{d}$ in lowest terms. The *mediant*[3] of $\frac{a}{b}$ and $\frac{c}{d}$, denoted $\frac{a}{b} \oplus \frac{c}{d}$, is the quotient

$$\frac{a}{b} \oplus \frac{c}{d} := \frac{a+c}{b+d}. \tag{2.1.1}$$

Remark. In general, we typically define the mediant of two fractions *already expressed in lowest terms.* Otherwise, the mediant would not be well-defined: from the formula alone, we would have $\frac{1}{2} \oplus \frac{3}{5} = \frac{4}{7}$ and $\frac{3}{6} \oplus \frac{12}{20} = \frac{15}{26}$, but $\frac{4}{7} \neq \frac{15}{26}$.

Example 2.1.3. Consider the fractions $\frac{a}{b} := \frac{3}{5}$ and $\frac{c}{d} := \frac{2}{3}$. Then these fractions are indeed in lowest terms, so

$$\begin{aligned} \frac{3}{5} \oplus \frac{2}{3} &= \frac{3+2}{5+3} \\ &= \boxed{\frac{5}{8}}. \end{aligned}$$

2.2 Exercises

2.2.1 Compute the following mediants:

(a) $\frac{1}{4} \oplus \frac{1}{3}$.

(b) $\frac{3}{5} \oplus \frac{2}{3}$.

[3] The mediant is also sometimes called the *Farey sum* of these rationals. It also sometimes is described as the "freshman sum", indicating that a common mistake newcomers make when learning how to add fractions is to assume—incorrectly—that in general, $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$.

(c) $\frac{6}{9} \oplus \frac{6}{8}$.

(d) In the definition of $\frac{a}{b} \oplus \frac{c}{d}$ above, the formula requires that our given pair of fractions must be in lowest terms. Provide an example to show that even when $\frac{a}{b}, \frac{c}{d}$ are in lowest terms, their mediant $\frac{a}{b} \oplus \frac{c}{d}$ need not be.
That is, provide an example of a pair of lowest-terms fractions $\frac{a}{b}, \frac{c}{d}$ such that the formula $\frac{a+c}{b+d}$ is *not* in lowest terms.

2.2.2 (a) Consider $\mathcal{F}_2$, the Farey sequence of order 2 as given in Equation (1.1.2) from Example 1.1.2: $\frac{0}{1}, \frac{1}{2}, \frac{1}{1}$. Compute the mediant of the outside terms, $\frac{0}{1} \oplus \frac{1}{1}$.

(b) Consider $\mathcal{F}_3$, the Farey sequence of order 3: $\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}$. Compute the mediants of the "almost neighbors": $\frac{0}{1} \oplus \frac{1}{2}, \frac{1}{3} \oplus \frac{2}{3}, \frac{1}{2} \oplus \frac{1}{1}$. What do you notice?

(c) Using your answers to Exercises #1.2.1–1.2.2, compute the mediants of these almost neighbors in $\mathcal{F}_4$. Do the same for $\mathcal{F}_5$ and for $\mathcal{F}_6$.

2.2.3 Consider $\mathcal{F}_5 = \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$.

(a) For each pair of consecutive lowest-terms fractions $\frac{a}{b} < \frac{c}{d}$ in $\mathcal{F}_5$, compute the value $bc - ad$. How does this relate to computing the distance $\frac{c}{d} - \frac{a}{b}$ between these terms?

(b) Consider the sum

$$\frac{1}{1\cdot 5} + \frac{1}{5\cdot 4} + \frac{1}{4\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 2} + \frac{1}{2\cdot 5} + \frac{1}{5\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 1}; \qquad (2.2.1)$$

that is, we sum over the reciprocals of the products of the denominators of consecutive terms in $\mathcal{F}_5$.

Hint: Computing the sum directly will likely be tedious, especially by hand. Using previous results, do you see a potential shortcut?

(c) Let n be a positive integer, and $\mathcal{F}_n$ the sequence

$$0 = \frac{a_1}{b_1} < \frac{a_2}{b_2} < \dots \frac{a_{|\mathcal{F}_n|}}{b_{|\mathcal{F}_n|}} = 1,$$

all in lowest terms. Prove that

$$\frac{1}{b_1 \cdot b_2} + \frac{1}{b_2 \cdot b_3} + \cdots + \frac{1}{b_{|\mathcal{F}_n|-1} \cdot b_{|\mathcal{F}_n|}} = 1. \qquad (2.2.2)$$

For our purposes here, you may use other results even if you haven't yet proven them. In particular, consider Theorem 2.2.1 in Exercise #2.2.6.

2.2.4 Let a, b, c, d be positive integers. Prove that if $\frac{a}{b} < \frac{c}{d}$, then

$$\frac{a}{b} < \frac{a}{b} \oplus \frac{c}{d} < \frac{c}{d}; \tag{2.2.3}$$

that is, prove both

$$\frac{a}{b} < \frac{a+c}{b+d} \quad \text{and} \quad \frac{a+c}{b+d} < \frac{c}{d}. \tag{2.2.4}$$

2.2.5 Let n be a positive integer with $n > 1$. Prove that no two consecutive terms in $\mathcal{F}_n$ have the same denominator. That is, if $\frac{a}{b}, \frac{c}{d}$ are consecutive lowest-terms entries in $\mathcal{F}_n$ with $n \geq 2$, then $b \neq d$.

Hint: Assume, to the contrary, that we have two consecutive lowest-terms fractions in $\mathcal{F}_n$, $\frac{a}{b}$ and $\frac{a+1}{b}$. In terms of a and b, can you insert a term between these values that also lies in $\mathcal{F}_n$?

Remark. This appears as Theorem 31 on page 24 of [3]

2.2.6 Consider the following two statements:

Theorem 2.2.1. *If $\frac{a}{b}$ and $\frac{c}{d}$ are two successive terms of $\mathcal{F}_n$, then*

$$bc - ad = 1. \tag{2.2.5}$$

Theorem 2.2.2. *If $\frac{a}{b}$, $\frac{e}{f}$, and $\frac{c}{d}$ are three successive terms of $\mathcal{F}_n$, then*

$$\frac{e}{f} = \frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}. \tag{2.2.6}$$

Prove that Theorem 2.2.1 is true if and only if Theorem 2.2.2 is true.

Remark. These theorems appear as Theorem 28 and 29, respectively, in [3]. Their equivalence is established on page 24 there.

2.2.7 Show that if $a/b, c/d$ are consecutive terms in $\mathcal{F}_n$, and in lowest terms, then $b+d > n$.

Note: For the purpose of this exercise, you may assume that Theorem 2.2.2 is true, whether or not you have proven it yet.

3 Geometric Tools and Pick's Theorem

3.1 Discussion

Remarkably, geometric methods can be incredibly useful in the context of number theoretic objects like the Farey sequence. Here are two useful theorems for how to compute areas:

Proposition 3.1.1. *If* $\mathbf{v}_1 := (a,b), \mathbf{v}_2 := (c,d) \in \mathbb{R}^2$, *then the area enclosed by the parallelogram determined by* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *is*

$$|\det(\mathbf{v}_1, \mathbf{v}_2)| := \left|\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| = |ad - bc|. \tag{3.1.1}$$

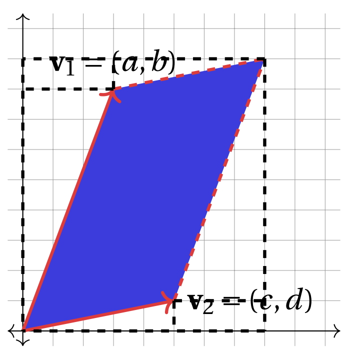

Figure 3.1.1: Computing the area of a parallelogram in the plane via determinants.

Here, det denotes the *determinant* of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For any such 2×2 matrix, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

Theorem 3.1.2 (Pick's Theorem)**.** *Let P be a simple polygon in the plane, each of whose vertices is a lattice point. (That is, for each vertex V on the boundary of P, both the x- and y-coordinates of V are integers.) Let I denote the number of lattice points in the interior of P, and let B denote the number of lattice points on the boundary. (I.e., B counts the number of lattice points that lie on the edges of the boundary of P.) Then the area A of P is given by the formula*

$$A = I + \frac{B}{2} - 1. \tag{3.1.2}$$

Remark. Proposition 3.1.1 and Pick's Theorem (Theorem 3.1.2) have each appeared in past Chapel Hill Math Circle sessions, including those for topics on continued fractions and Pick's Theorem itself.

3.2 Exercises

3.2.1 Consider Exercise #2.2.4, and view $\frac{a}{b}$, $\frac{c}{d}$ as the points (b,a) and (d,c), respectively, in the plane. Explain how we can view Equation (2.2.3) geometrically in the context of Figure 3.2.1:

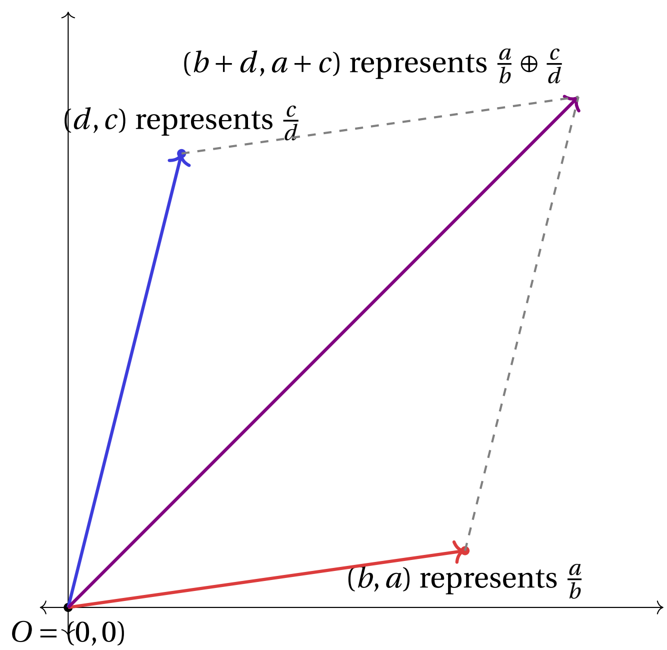

Figure 3.2.1: Interpreting the mediant $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$ as the vector sum $(b,a)+(d,c) = (b+d, a+c)$.

3.2.2 Prove Theorem 2.2.1 using Pick's Theorem (Theorem 3.1.2).

Hint: Let $a/b, c/d$ be consecutive terms in $\mathcal{F}_n$, and consider the parallelogram generated by (b, a) and (d, c). Compute its area two ways.

3.2.3 For a given fraction of the form $\frac{a}{b}$, consider the point (b, a) in the plane, so that the line through (b, a) and the origin $(0, 0)$ has slope $\frac{a}{b}$. For a positive integer n, draw the corresponding points from $\mathcal{F}_n$, and connect consecutive terms with straight line segments. For example, for $\mathcal{F}_3$, we form the following:

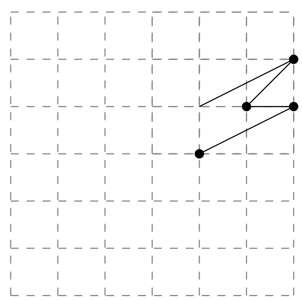

Figure 3.2.2: Visualizing $\mathcal{F}_3$ by identifying the fraction $\frac{a}{b}$ with the point (b, a).

We can then extend this symmetrically to form a sunburst pattern: Form the same

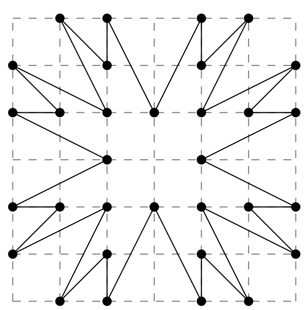

Figure 3.2.3: Complete "sunburst" polygon generated from $\mathcal{F}_3$.

sunburst representations of $\mathcal{F}_4$, $\mathcal{F}_5$, and $\mathcal{F}_6$.

3.2.4 Let $a/b, c/d$ be lowest-terms fractions with $0 \leq a/b, c/d \leq 1$. Prove that if $bc - ad = 1$, then $a/b, c/d$ are consecutive terms in some Farey sequence $\mathcal{F}_n$.

3.2.5 In Exercise #3.2.3, we introduced sunburst-shaped polygons associated with $\mathcal{F}_n$. Using Pick's Theorem, what is the area enclosed by the sunburst, as a function of n?

Hint: Exercise #1.2.6 will be useful here.

4 Ford Circles and Farey Sequences

4.1 Discussion

Definition 4.1.1. Let p, q be relatively prime integers with $p \neq 0$. The *Ford Circle at* $\frac{p}{q}$, denoted $C(\frac{p}{q})$, is the circle whose center is $\left(\frac{p}{q}, \frac{1}{2q^2}\right)$ and with radius $\frac{1}{2q^2}$.

Example 4.1.2. The Ford Circles at 0/1, 1/1, 1/2, 1/3, and 2/3 are illustrated in Figure 4.1.1:

4.2 Exercises

4.2.1 Prove that every Ford Circle $C\left(\frac{p}{q}\right)$ is tangent to the x-axis at the point $\left(\frac{p}{q}, 0\right)$.

4.2.2 Prove that if $\frac{a}{b}$, $\frac{c}{d}$ are any two distinct lowest-terms rational numbers, then $C\left(\frac{a}{b}\right)$ and $C\left(\frac{c}{d}\right)$ are either disjoint or tangent to each other.

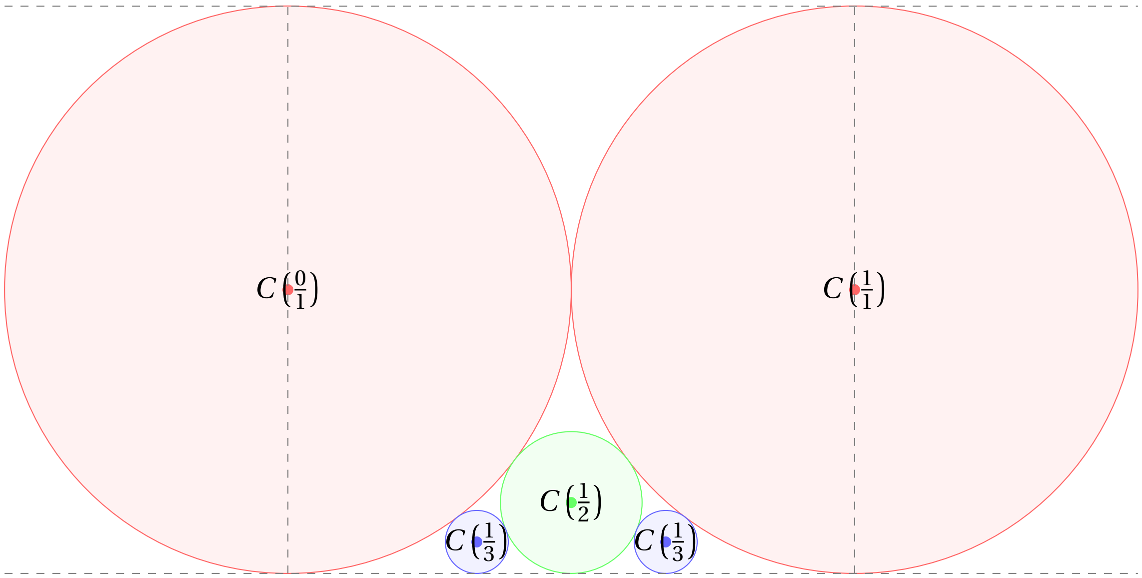

Figure 4.1.1: Showing the Ford circles $C(0/1), C(1/3), C(1/2), C(2/3), C(1/1)$.

4.2.3 Let $\frac{a}{b}, \frac{c}{d}$ be consecutive entries in $\mathcal{F}_n$ expressed in lowest terms, and assume that $C\left(\frac{a}{b}\right)$ and $C\left(\frac{c}{d}\right)$ are tangent to each other. (One can prove tangency for these circles, but for now we shall simply assume it.) Prove that the Ford Circle for the mediant $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$, $C\left(\frac{a+c}{b+d}\right)$, is tangent to $C\left(\frac{a}{b}\right)$ and $C\left(\frac{c}{d}\right)$.

4.2.4 Let $a/b, c/d$ be distinct lowest-terms fractions, with $0 \leq a/b, c/d \leq 1$. Prove that the Ford Circles $C\left(\frac{a}{b}\right)$ and $C\left(\frac{c}{d}\right)$ are tangent if and only if $a/b, c/d$ are consecutive terms in some Farey sequence $\mathcal{F}_n$.

5 Additional Comments and Exercises

5.1 Discussion

The exercises in the following subsection will include a few that are a bit more challenging. Before proceeding to them, it might be worth mentioning a few applications I discovered:

- The *Stern–Brocot tree*[4] is closely related to the Farey sequence. Like our sequence, the Stern–Brocot tree is formed by taking mediants.
- Taking mediants, and therefore the Farey sequence and Stern–Brocot tree, has relevance to clockmaking. For some explanation of how and why, see Section 2.3 of [2].
- Farey sequences have applications in the theory of *rational approximations* to real numbers, which ask for whether we can find "good" rational approximations to a given irrational number, whether there are finitely or infinitely many such "good" rational approximations, and if any exist, how we might find them. One example that can be proven by Farey sequences is the following theorem by Dirichlet:[5]

  Theorem 5.1.1 (Dirichlet's Approximation Theorem)**.** *Let α, N be real numbers with $N \geq 1$. Then there exist integers p, q with $1 \leq q \leq N$ such that*

  $$\left|q\alpha - p\right| \leq \frac{1}{\lfloor N\rfloor + 1} < \frac{1}{N}. \tag{5.1.1}$$

  A slightly weaker version says the following: if α is any real number and N is a positive integer, then there exist integers p, q with $1 \leq q \leq N$ such that

  $$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q(N+1)}, \tag{5.1.2}$$

  from which it follows that for any irrational number α, there are infinitely many fractions $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$.
- Whenever you are exploring a number theory topic involving fractions, it is worth considering them in the context of *continued fractions*. There are certainly a number of interesting properties about the Farey in terms of continued fraction representations. For example, there appear to be interesting properties about the continued fraction representation of $\frac{a}{b}$ and those of its immediate neighbors in $\mathcal{F}_n$, though exploring them lies beyond the scope of this session.

[4]This tree is named for German mathematician Moritz Abraham Stern (1807–1894) and French clockmaker Achille Brocot (1817–1878).

[5]Peter Gustav Lejeune Dirichlet (1805–1859) was a German mathematician.

5.2 Exercises

5.2.1 **Challenging:** Let n be a positive integer. If $\frac{a}{b} < \frac{c}{d}$ are consecutive lowest-terms fractions in $\mathcal{F}_n$, if $\frac{c}{d} < 1$, how can you compute the next term $\frac{p}{q}$ in $\mathcal{F}_n$?

5.2.2 **Very Challenging:** Let n be a positive integer. If the lowest-terms fraction $\frac{a}{b}$ lies in $\mathcal{F}_n$, and $\frac{a}{b} \neq 0, 1$, can you find the next element $\frac{p}{q} > \frac{a}{b}$ in $\mathcal{F}_n$?

For example, following [7], consider $\mathcal{F}_{234}$. Since $217 \leq 234$, the term $\frac{a}{b} := \frac{143}{217}$ lies in $\mathcal{F}_{234}$. What is the term $\frac{p}{q}$ in $\mathcal{F}_{234}$ immediately following $\frac{143}{217}$?

Note: One possible solution to this likely uses certain ideas beyond the scope of this session, such as the complete parametrization of all the solutions to a *linear Diophantine equation*, seeking all *integer* solutions (x, y) to an equation of the form $ax + by = c$.

5.2.3 **2011 American Mathematics Competition (AMC) 10B, Problem 24 (Challenging):** "A lattice point in an xy-coordinate system is any point (x, y) where both x and y are integers. The graph of $y = mx + 2$ passes through no lattice point with $0 < x \leq 100$ for all m such that $\frac{1}{2} < m < a$. What is the maximum possible value of a?

(A) $\frac{51}{101}$ **(B)** $\frac{50}{99}$ **(C)** $\frac{51}{100}$ **(D)** $\frac{52}{101}$ **(E)** $\frac{13}{25}$"

References

[1] Abide By Reason. Ford circles and Farey sequences. https://www.youtube.com/watch?v=wE1iJkXs1Xg, February 7, 2025.

[2] Jonathan Ainsworth, Michael Dawson, John Pianta, and James Warwick. The Farey sequence. https://webhomes.maths.ed.ac.uk/~v1ranick/fareyproject.pdf, March 15, 2012. online: retrieved February 24, 2026.

[3] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press, Walton Street, Oxford OX2 6DP, fifth edition, 1979.

[4] Dr. Loveless Teaches Math. Farey sequence facts, history and Ford circles. https://www.youtube.com/watch?v=s-2GWE5UbLw, March 25, 2025.

[5] Dr. Loveless Teaches Math. Farey sequences - an application of adding fractions incorrectly. https://www.youtube.com/watch?v=UaO69Set66g, March 25, 2025.

[6] Ivan Niven, Herbert S. Zuckerman, and Hugh L. Montgomery. *An Introduction to the Theory of Numbers*. John Wiley & Sons, Inc., New York, fifth edition, 1991.

[7] Numberphile. Funny fractions and Ford circles - Numberphile. https://www.youtube.com/watch?v=0hlvhQZIOQw, June 9, 2015.

[8] vEnhance (Evan Chen). USAMO 1999/3: Introducing FAREY SEQUENCES! https://www.youtube.com/watch?v=LNBcuBMmN9g, November 13, 2020.

[9] zhuli. What happens if we add fractions incorrectly? #SoME3. https://www.youtube.com/watch?v=4d6YrTKmjfE, August 1, 2023.

[10] Dylan Zukin. The Farey sequence and its niche(s). https://www.whitman.edu/Documents/Academics/Mathematics/2016/Zukin.pdf, May 12, 2016. online: retrieved December 15, 2022.