CHAPEL HILL MATH CIRCLE EXIT TICKET:
January 17, 2026: Thinking Combinatorially

Please remove this sheet, complete it, and return it before the end of our session.

- Did you find today's topic interesting?
- Was the this topic appropriately challenging relative to your background? That is, was the topic neither too elementary nor inaccessibly advanced?
- How could we improve this worksheet for future sessions?
- What did you enjoy about today's topic?
- What did you find particularly challenging?
- Was there anything you thought was too difficult?
- Was there anything you thought was too easy?
- Are there any topics you would be interested in seeing us cover in the future?

Thinking Combinatorially

Abstract

A number of algebraic identities arise from the branch of mathematics called *combinatorics*. In this session, we shall prove a number of algebraic identities by using combinatorial methods rather than other tools such as mathematical induction or more straightforward algebra. This is a new version of the session for Chapel Hill Math Circle's advanced group on March 19, 2022, now including many exercises taken from mathematics competitions.

Background needed: Prerequisites include basic algebra, ideally including some familiarity with some introductory concepts from combinatorics. That said, most of this worksheet should be otherwise self-contained.

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Contents

0 Warmup

As prerequisites for this session, it will help to answer the following first. You do *not* need to know these answers already, and many questions will be revisited later in the worksheet.

0.1 Let S be a set. What does it mean to say that the set T is a *subset* of S, denoted $T \subseteq S$? What is the *empty set*, denoted $\varnothing$? What is the *power set* of S, denoted $\mathcal{P}(S)$? List all the subsets of $\{1,2,3\}$, as well as the power set of $\{1,2,3\}$.

0.2 Let n be a nonnegative integer. What is $n!$, read as *n factorial*?

0.3 Let n, k be nonnegative integers. What is *n choose k*? Does your answer depend on whether k or n is the larger number?

Notation. Typically, n choose k is denoted

$$\binom{n}{k}, \tag{0.0.1}$$

with other notation like C^n_k, ${}_nC_k$, and $C(n,k)$ also common. The values $\binom{n}{k}$ are also called *binomial coefficients*, for reasons that will make more sense after exploring the Binomial Theorem (Theorem 3.1.1).

0.4 What is *Pascal's Triangle*?

Remark. Pascal's Triangle was the topic for the advanced group, most recently on both September 13, 2025 and September 27, 2025.

1 Counting the Subsets of Finite Sets

1.1 Discussion

To begin, let us use the following convention to simplify notation:

Notation. Let n be a positive integer. Then we set

$$[\mathbf{n}] := \{1, 2, \ldots, n\}. \tag{1.1.1}$$

and $[\mathbf{0}] := \varnothing$.

Next, we provide an answer to Exercise #0.3:

Definition 1.1.1. Let n, k be nonnegative integers. Then n *choose* k, denoted $\binom{n}{k}$, is the number of distinct k-element subsets of an n-element set.

The following result is absolutely fundamental in combinatorics:

Theorem 1.1.2. *Let n, k be nonnegative integers with $k \leq n$. By definition the number of distinct k-element subsets of an n-element set is given by $\binom{n}{k}$, and*

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} \tag{1.1.2}$$

You are encouraged to try to *prove* Theorem 1.1.2, but our primary focus for now is being able to use this conceptual understanding of what $\binom{n}{k}$ *represents* in order to prove combinatorial identities.

Example 1.1.3. Consider the case $n := 4$, $k := 2$. Then by Theorem 1.1.2, there are precisely $\binom{4}{2} = \frac{4!}{2! \cdot 2!} = 6$ subsets of a 4-element set that contain precisely 2 elements.

For example, if $S := [\mathbf{4}] = \{1, 2, 3, 4\}$, then the 2-element subsets of S are precisely $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, and $\{3, 4\}$. We therefore see there are indeed 6 such 2-element subsets.

One of the ways Theorem 1.1.2 becomes especially useful is through the following:

Strategy 1.1.4. *Algebraic identities can often be verified by counting a particular set in two different ways.*

Proposition 1.1.5 (Pascal's Rule)**.** *Let n, k be nonnegative integers. Then*

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}. \tag{1.1.3}$$

Remark. Compare Proposition 1.1.5 with Warmup Exercise #0.4. What can we conclude?

One can prove (1.1.3) a number of algebraic ways, such as using mathematical induction or via the formula in (1.1.2). Instead, we illustrate Strategy 1.1.4.

Proof. Viewing Pascal's Rule conceptually, the claim is that

the number of k-element subsets of an $(n+1)$-element set
= the sum of the number of k-element subsets of an n-element set
+
the number of $(k-1)$-element subsets of an n-element set. (1.1.4)

Without loss of generality, set $S := [\mathbf{n+1}]$, so $|S| = n+1$. Consider any subset T of S having precisely k elements. There are two possibilities:

Case 1: $n+1 \notin T$.

Then since $n+1 \notin T$, T is thus a k-element subset of $[\mathbf{n}]$. Any such T is therefore a k-element subset of the n-element set $[\mathbf{n}] = S \setminus \{n+1\}$, and by Theorem 1.1.2, there are precisely $\binom{n}{k}$ such sets of this form.

Case 2: $n+1 \in T$.

Since $n+1 \in T$, we have that T is of the form $T' \cup \{n+1\}$, where T' is a $(k-1)$-element subset of $[\mathbf{n}]$. As in Case 1, there are $\binom{n}{k-1}$ such subsets T', and thus $\binom{n}{k-1}$ such subsets T of S for which $n+1 \in T$.

The total number of k-element subsets of S is therefore the sum from these two cases, whence

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \tag{1.1.5}$$

which verifies (1.1.3). □

By Pascal's Rule, we see that the kth element in the nth row of Pascal's Triangle, beginning at the top with row $n = 0$, is $\binom{n}{k}$. □

Example 1.1.6. In Example 1.1.3, we used the formula in Theorem 1.1.2 to compute $\binom{4}{2}$. Here, we use Pascal's Rule,

According to Pascal's Rule, $\binom{4}{2} = \binom{3}{2} + \binom{3}{1}$. Continuing,

$$\begin{aligned}
\binom{4}{2} &= \binom{3}{2} + \binom{3}{1} \\
&= \left[\binom{2}{2} + \binom{2}{1}\right] + \left[\binom{2}{1} + \binom{2}{0}\right] \\
&= \binom{2}{2} + 2\cdot\binom{2}{1} + \binom{2}{0} \\
&= \binom{2}{2} + 2\cdot\left[\binom{1}{1} + \binom{1}{0}\right] + \binom{2}{0} \\
&= \binom{2}{2} + \binom{2}{0} + 2\cdot\binom{1}{1} + 2\cdot 10 \\
&= 1 + 1 + 2\cdot 1 + 2\cdot 1 \\
&= 1 + 1 + 2 + 2 \\
\Rightarrow \binom{4}{2} &= 6.
\end{aligned}$$

1.2 Exercises

1.2.1 Since one of the core ideas of this session is that of the binomial coefficient *n choose k*, denoted $\binom{n}{k}$. Here, $\binom{n}{k}$ counts the number of distinct k-element subset of a n-element set.

(a) Enumerate all distinct 2-element subsets of $\{1,2,3,4,5\}$. Use this to compute $\binom{5}{2}$.

(b) Enumerate all distinct 3-element subsets of $\{1,2,3,4,5\}$. Use this to compute $\binom{5}{3}$.
Hint: Can we simplify our work by using the result from Exercise #1.2.1(a)?

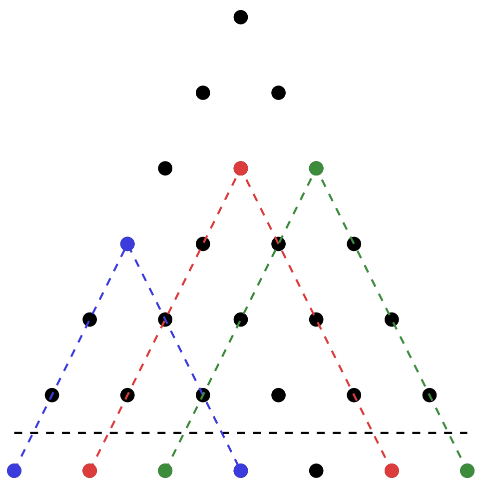

Figure 1.2.1: Illustrating that $1+2+3+\cdots+n = \binom{n+1}{2}$ for the specific case $n = 5$. Any point in the first n rows determines a unique pair of distinct points in row $n+1$, and every pair of points in row $n+1$ determines a unique point in the top n rows.

1.2.2 Let n be a positive integer. Prove that

$$1+2+3+\cdots+n = \binom{n+1}{2}. \tag{1.2.1}$$

Hint: Think geometrically. The left-hand side represents the total number of elements in an equilateral triangular array for the nth triangular number, as in Figure 1.2.1. Can you produce a one-to-one correspondence from each such point to a unique pair among a set with $n+1$ elements?

1.2.3 Let m, n be positive integers. Prove

$$\binom{n+m}{2} = \binom{n}{2} + \binom{m}{2} + mn. \tag{1.2.2}$$

Hint: How many 2-elements subsets are there of $[\mathbf{n}+\mathbf{m}]$? Viewing $[\mathbf{n}+\mathbf{m}]$ as the set

$$[\mathbf{m}+\mathbf{n}] = \{\underbrace{1, 2, \cdots, m,}_{m \text{ elements}}\ \underbrace{m+1, m+2, \ldots, m+n}_{n \text{ elements}}\}, \tag{1.2.3}$$

compute the number of 2-element subsets in two different ways.

1.2.4 Let n be a nonnegative integer. Prove that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}+\binom{n}{n}=2^n. \tag{1.2.4}$$

Remark. Note that the left-hand side of (1.2.4) is the sum of the numbers of 0-element subsets of [**n**], of 1-element subsets of [**n**], up to the number of n-element subsets of [**n**]. Together, this counts the *total* number of subsets of [**n**]. This exercise is equivalent to asking the following: if S is a set with precisely n elements, what is the size of $\mathcal{P}(S)$, the *power set* of S? The power set was introduced in Exercise #0.1.

1.2.5 Let n, k be nonnegative integers with $1 \leq k \leq n$. Prove that

$$k\cdot\binom{n}{k}=n\cdot\binom{n-1}{k-1}. \tag{1.2.5}$$

<u>*Hint:*</u> How many ways are there to select a k-element committees, each with a single president, from among n-element group of people?

1.2.6 Prove Theorem 1.1.2: if n, k are nonnegative integers, and $k \leq n$, (1.1.2) holds:

$$\binom{n}{k}=\frac{n!}{k!\cdot(n-k)!}.$$

1.2.7 Let n, k, and ℓ be nonnegative integers such that $\ell \leq k \leq n$. Prove that

$$\binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}\binom{n-\ell}{k-\ell}. \tag{1.2.6}$$

Hint: Starting with a class of n people, form a committee with k members. From this committee, form a subcommittee of ℓ members. Can you provide two different ways to count how to form such a committee with subcommittee?

Remark. Note that Exercise #1.2.5 is a special case of Equation (1.2.6) for the case $\ell = 1$.

1.2.8 Let n be a positive integer. Prove that

$$\binom{n}{1} + 2\cdot\binom{n}{2} + 3\cdot\binom{n}{3} + \cdots + n\cdot\binom{n}{n} = n\cdot 2^{n-1}. \tag{1.2.7}$$

Remark. This exercise admits a combinatorial solution, which is the method to practice here. For those of you who have studied calculus, there is an alternate approach: using the Binomial Theorem (Theorem 3.1.1) expansion for $(x+1)^n$, one can also prove the identity in (1.2.7) via differentiation. Namely, express $(x+1)^n$ in two ways, differentiate each with respect to x, then evaluate both sides of your equation at $x = 1$.

1.2.9 Prove *Vandermonde's Identity*:[1] If m, n, and k are nonnegative integers with $k \leq$

[1]Vandermonde's Identity is the namesake of Alexandre-Théophile Vandermonde (1735–1796), a French mathematician, musician, and chemist.

$m+n$, then

$$\binom{m+n}{k} = \binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \binom{m}{2}\binom{n}{k-2} + \cdots + \binom{m}{k}\binom{n}{0}$$
$$= \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}. \tag{1.2.8}$$

Hint: For the original statement of Vandermonde's Identity, imagine a class with m ninth graders and n tenth graders, and we seek to form a committee with k students drawn from across the two grades. Count the number of such possible committees in two different ways to establish (1.2.8).

Remark. Vandermonde's Identity is often used to simplify solutions to combinatorics exercises that appear in mathematics competitions. See Section 5 for examples.

As a generalization of Vandermonde's Identity, if $n_1, n_1, \ldots, n_r, k$ are nonnegative integers and r is a positive integer, then

$$\binom{n_1+n_2+\cdots+n_r}{k} = \sum_{\substack{0\le k_1,k_2,\ldots,k_r\le k\\ k_1+k_2+\cdots+k_r=k}} \binom{n_1}{k_1}\cdot\binom{n_2}{k_2}\cdot\cdots\cdot\binom{n_r}{k_r}. \tag{1.2.9}$$

That is, the right-hand side of (1.2.9) is summing over all products of r binomial coefficients $\binom{n_j}{k_j}$, where the k_j are nonnegative integers satisfying $k_1+k_2+\cdots+k_r = k$. (The original Vandermonde's Identity in (1.2.8) is the specific instance of (1.2.9) for the case $r=2$.)

1.2.10 Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \tag{1.2.10}$$

2 Counting with Bijections

2.1 Discussion

We begin with a technical definition, something previously discussed in recent advanced group sessions of Chapel Hill Math Circle:

Definition 2.1.1. Let $f\colon S \to T$ be a function between sets S and T. We say that f is a *bijection* (or a *one-to-one correspondence*) if and only if

- f is an *injection* or *one-to-one*: if $f(s) = f(s')$, then $s = s'$; equivalently, if $s \neq s'$, then $f(s) \neq f(s')$

  and

- f is *surjective* or onto: for every $t \in T$, there exists some $s \in S$ such that $f(s) = t$.

That is, in a bijection $f\colon S \to T$, every element in T gets touched by f, and each gets touched by precisely one element in S.

It should be intuitively clear that two finite sets S and T are the same size if and only if[2] there is a bijection $f\colon S \to T$. We can therefore reformulate Strategy 1.1.4 as follows:

Strategy 2.1.2. *Let S, T be finite sets. To show that $|S| = |T|$, it suffices to produce a bijection $f\colon S \to T$.*

That is, a bijection effectively is *a way of counting a finite set in two different ways!*

Example 2.1.3. Let us prove that $\binom{4}{1} = \binom{4}{3}$ using a bijection. Let

$$S := \text{the set of all 1-element subsets of } \{1,2,3,4\}$$
$$T := \text{the set of all 3-element subsets of } \{1,2,3,4\}.$$

Then consider the map defined as follows:

$$f\colon S \to T$$
$$f(A) := \{1,2,3,4\} \setminus S.$$

That is, f takes a 1-element subset A, and $f(A)$ is the set of all elements in $\{1,2,3,4\}$ that are *not* in A. (For example, $\{2\}$ is a 1-element subset of $\{1,2,3,4\}$, and $f(A) := \{1,3,4\}$, the set of all elements *other than* 2.)

[2] For technical reasons, this remains true even when $S = T = \varnothing$.

2.2 Exercises

2.2.1 Let n, k be nonnegative integers with $0 \le k \le n$. Prove that

$$\binom{n}{k} = \binom{n}{n-k}. \tag{2.2.1}$$

Hint: Consider this in the context of Example 2.1.3.

2.2.2 Let n be a positive integer. Prove that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots \tag{2.2.2}$$

Combining this with (1.2.4) from Exercise #1.2.4, we therefore have that for all $n \ge 1$,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}. \tag{2.2.3}$$

To elaborate, if n is a positive integer, then the left-hand size of (2.2.2) counts the number of subsets of $[\mathbf{n}]$ of size $0, 2, 4, \ldots$, or all subsets of even size. Similarly, the right-hand size of (2.2.2) counts the number of subsets of $[\mathbf{n}]$ of size $1, 3, 5, \ldots$, or all subsets of odd size. Equation (2.2.2) therefore means that there are as many even-sized subsets of $[\mathbf{n}]$ as there are odd-sized subsets of $[\mathbf{n}]$. Further, for $n \ge 1$,[3] the total number of subsets of either parity is 2^{n-1}.

Remark. The identity in (2.2.3) can, perhaps unexpectedly, be used in a proof of the *Pythagorean Theorem* of all things! For details, see [11].

[3] The case $n = 0$ is exceptional: $[\mathbf{0}] = \varnothing$ has the unique subset $\varnothing$, so there are *no* subsets of $\varnothing$ that have odd size.

3 The Binomial Theorem

3.1 Discussion

Let x, y be indeterminates. If n is a positive integer, what can we say about $(x+y)^n$? The answer is given by the *Binomial Theorem*:

Theorem 3.1.1 (The Binomial Theorem)**.** *Let x, y be indeterminates, and let n be a positive integer. Then*

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{0}y^n. \tag{3.1.1}$$

Remark. In *sigma notation*, Equation (3.1.1) can be rewritten as

$$(x+y)^n = \sum_{j-0}^{n} \binom{n}{j} x^{n-j} y^j. \tag{3.1.2}$$

Example 3.1.2. Let us verify the Binomial Theorem holds when $n = 2$. We have

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= x^2 + xy + xy + y^2\\ &= x^2 + 2xy + y^2\\ &= \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2,\end{aligned}$$

as desired.

3.2 Exercises

3.2.1 Verify the Binomial Theorem for $n = 3$. That is, show that

$$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3. \tag{3.2.1}$$

3.2.2 Expand $(x+y)^4$ and $(x+y)^5$. What can you say about the coefficients in these expansions?

3.2.3 Prove the Binomial Theorem (Theorem 3.1.1).

Note: Any valid proof is acceptable, but try to find a combinatorial argument.

3.2.4 Prove Exercise #1.2.4 using The Binomial Theorem.

3.2.5 Prove Exercise #2.2.2 using The Binomial Theorem.

3.2.6 Exercise #1.2.9 states Vandermonde's Identity: If m, n, and k are nonnegative integers with $k \leq m+n$, then

$$\binom{m+n}{k} = \binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \binom{m}{2}\binom{n}{k-2} + \cdots + \binom{m}{k}\binom{n}{0}. \tag{3.2.2}$$

Prove Vandermonde's Identity using the Binomial Theorem.

Hint: The left-hand side of (3.2.2) is the coefficient of x^k in the expansion of $(x+1)^{m+n}$. We also have that $(x+1)^{m+n} = (x+1)^m \cdot (x+1)^n$. What can you conclude?

3.2.7 **Challenging:** In the Binomial Theorem (Theorem 3.1.1 in Exercise #3.2.3), we have a formula for taking the binomial term $x+y$, then computing its powers in terms of powers of x and y. Can you generalize this to expanding expressions of the form $(x+y+z)^n$, where n is a nonnegative integer?

For example, we have

$$(x+y+z)^1 = x^1 + y^1 + z^1$$
$$(x+y+z)^2 = x^2 + 2xy + y^2 + 2yz + z^2 + 2xz$$
$$(x+y+z)^3 = x^3 + 3x^2y + 3xy^2 + y^3 + 3y^2z + 3yz^3 + z^3 + 3xz^2 + 3z^2z + 6xyz.$$

Can you explain how to compute these coefficients in a manner similar to that given by the binomial theorem? Can you generalize this to sums of the form $(x+y+z)^n$? What about $(x+y+z+w)^n$? What about more general expansions of the form

$$(x_1 + x_2 + \cdots + x_m)^n, \tag{3.2.3}$$

where $m \geq 2$ is a positive integer, and n is a nonnegative integer?

Remark. This exercise is effectively asking you to conjecture, state carefully, and if possible prove *The Multinomial Theorem*, which explains how to compute expansions of the form in (3.2.3).

Further, note that in particular, the Binomial Theorem is the reduction of The Multinomial Theorem to when $m = 2$, so your formulation of the Multinomial Theorem must agree with the Binomial Theorem in this case.

4 Additional Exercises

4.1 Discussion

In this section, we ask additional questions to practice combinatorial approaches to verifying identities and simplifying expressions.

4.2 Exercises

4.2.1 Let n be a nonnegative integer. Find a simple expression for the sum

$$\sum_{k=0}^{n} k(k-1)\binom{n}{k} = 0\cdot(-1)\cdot\binom{n}{0} + 1\cdot 0\cdot\binom{n}{1} + 2\cdot 1\cdot\binom{n}{2} + \cdots + n(n-1)\binom{n}{n}, \qquad (4.2.1)$$

then provide a combinatorial proof of your result. Can you generalize your result?

Hint: Compare to Exercise #1.2.8.

Note: This appears as Problem 5 in [18].

4.2.2 Let n be a nonnegative integer. Find a simple expression for the sum

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{1}{0+1}\cdot\binom{n}{0} + \frac{1}{1+1}\cdot\binom{n}{1} + \frac{1}{2+1}\cdot\binom{n}{2} + \cdots + \frac{1}{n+1}\cdot\binom{n}{n}. \qquad (4.2.2)$$

Hint: Prove that for all nonnegative integers n, k with $k \le n$,

$$\frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k+1}. \qquad (4.2.3)$$

Using this, rewrite the sum (4.2.2).

Note: This appears as Problem 6 in [18]. Further, as in Exercise #1.2.8, there is a calculus-based approach to simplifying the above sum.

4.2.3 Simplify the following sums:

(a)

$$\binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{2} + \cdots + \binom{n}{n-1}\cdot\binom{n}{n}$$

(b)

$$\binom{n}{0}\binom{n}{2}+\binom{n}{1}\binom{n}{3}+\cdots+\binom{n}{n-2}\cdot\binom{n}{n}$$

(c)

$$\binom{n}{0}\binom{n}{r}+\binom{n}{1}\binom{n}{r+1}+\cdots+\binom{n}{n-r}\cdot\binom{n}{n},$$

where $0 \le r \le n$.

Note: This appears as Honors Problem IV in [18]—which states that these identities arose from acoustics!

4.2.4 Let n, m be nonegative integers with $m \le n$. Prove the *Hockey Stick Identity*:[4]

$$\binom{0}{k}+\binom{1}{k}+\binom{2}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}. \tag{4.2.4}$$

Note: recall that by our definition of $\binom{n}{k}$, $\binom{m}{k}=0$ when $m<k$.

Remark. The Hockey Stick Identity is often used to simplify solutions to combinatorics exercises that appear in mathematics competitions. See Section 5 for examples.

[4]This gets its athletic name by locating each of the above elements in Pascal's Triangle. The resulting shape resembles a hockey stick.

4.2.5 Let n be a nonnegative integer. Prove that

$$n^2 = 2 \cdot \binom{n}{2} + n. \tag{4.2.5}$$

4.2.6 Let n be a nonnegative integer. Prove that

$$\binom{2n+2}{n+1} = \binom{2n}{n+1} + 2 \cdot \binom{2n}{n} + \binom{2n}{n-1}. \tag{4.2.6}$$

4.2.7 Let n be a positve integer. Prove that

$$1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1 = \binom{n+2}{3}. \tag{4.2.7}$$

4.2.8 **Challenging:** Prove that if $n \geq 4$, then

$$\binom{\binom{n}{2}}{2} = 3 \cdot \binom{n+1}{4}. \tag{4.2.8}$$

Hint: Let S_n be a set of n equally spaced points lying on a common circle. The left-hand side of (4.2.8) counts the number of pairs of *pairs* of points from S_n, or pairs of line segments with endpoints from S_n. How large is this set of pairs of pairs of points? Can you think of a second interpretation of our set to complete a proof of this identity?

Remark. From Equation (1.1.2),

$$\binom{\binom{n}{2}}{2} = \frac{\left(\frac{n(n-1)}{2}\right)!}{2! \cdot \left(\frac{n(n-1)}{2} - 2\right)!}.$$

This seems complicated to compute algebraically, further motivating a combinatorial approach.

This exercise is taken as the first identity proven in [16]. Those who are interested might also try to prove some of the other identities presented there:

$$\binom{\binom{n}{2}}{3} = 15 \cdot \binom{n}{6} + 30 \cdot \binom{n}{5} + 16\binom{n}{4} + \binom{n}{3};$$
$$\binom{\binom{n}{3}}{2} = 10 \cdot \binom{n}{6} + 15 \cdot \binom{n}{5} + 6\binom{n}{4},$$
$$\binom{\binom{n}{4}}{2} = 35 \cdot \binom{n}{8} + 70 \cdot \binom{n}{7} + 45 \cdot \binom{n}{6} + 10 \cdot \binom{n}{5},$$

and culminating in a general case

$$\binom{\binom{n}{k}}{2} = \frac{1}{2} \cdot \sum_{j=1}^{k-1} \binom{n}{2k-j}\binom{2k-j}{j}\binom{2k-2j}{k-j}.$$

See also "A weight formula for subgraphs of K_n and log-concavity of nested binomial coefficients" on Math Overflow for additional identities involving expressions of the form $\binom{\binom{n}{k}}{m}$.

Separately, using Equation (4.2.8) gives us a method for solving Problem 10 from the 2022 AIME II: "Find the remainder when

$$\binom{\binom{3}{2}}{2} + \binom{\binom{4}{2}}{2} + \cdots + \binom{\binom{40}{2}}{2}$$

is divided by 1000."

5 Problems from Mathematics Competitions

5.1 Discussion

Techniques from the above sections can be used to solve a number of problems from various mathematics competitions, including the *American Mathematics Competitions (AMCs)*, the *American Invitational Mathematics Examination (AIME)*, the *International Mathematical Olympiad (IMO)*, and the *William Lowell Putnam Mathematical Competition ("the Putnam")*. Below is a sample of exercises from such contests.

5.2 Exercises

5.2.1 **1986 AIME, Problem 11:** "The polynomial $1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$ may be written in the form $a_0 + a_1 y + a_2 y^2 + \cdots + a_{16} y^{16} + a_{17} y^{17}$, where $y = x + 1$ and the a_i's are constants. Find the value of a_2."

5.2.2 **2000 AIME II, Problem 7:** "Given that

$$\frac{1}{2!17!} + \frac{1}{3!16!} + \frac{1}{4!15!} + \frac{1}{5!14!} + \frac{1}{6!13!} + \frac{1}{7!12!} + \frac{1}{8!11!} + \frac{1}{9!10!} = \frac{N}{1!18!} \tag{5.2.1}$$

find the greatest integer that is less than $\frac{N}{100}$."

Hint: Multiply both sides of (5.2.1) by 19!, so that the resulting expression becomes a sum of binomial coefficients. Can you use additional results from above to further simplify the new sum? From that, can you compute N, and using that, $\lfloor \frac{N}{100} \rfloor$?

5.2.3 **2020 AIME I, Problem #7**: "A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committe could have as few as 1 member or as many as 23 members. Let N be the number of such committees that can be formed. Find the sum of the prime numbers that divide N."

5.2.4 **Challenging:** The following two exercises are worth including as a pair:

(a) **2015 AIME I, Problem 12:** "Consider all 1000-element subsets of the set $\{1,2,3, ..., 2015\}$. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$."

(b) **1981 IMO, Problem 2:** "Let $1 \le r \le n$ and consider all subsets of r elements of the set $\{1,2,\ldots,n\}$. Each of these subsets has a smallest member. Let $F(n,r)$ denote the arithmetic mean of these smallest numbers; prove that

$$F(n,r) = \frac{n+1}{r+1}.\text{"} \tag{5.2.2}$$

Example 5.2.1. Consider the case $n = 3$, $r = 2$, so that we are asked to compute the arithmetic mean of all 2-element subsets of $\{1,2,3\}$. Our 2-element subsets are precisely $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$; their minimum values are, respectively, 1, 1, and 2. The arithmetic mean for this example is therefore

$$\frac{1+1+2}{3} = \frac{4}{3},$$

and this agrees with our formula in Equation (5.2.2) for this case.

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