Chapel Hill Math Circle Exit Ticket:
September 27, 2025: Pascal's Triangle

Please remove this sheet, complete it, and return it before the end of our session.

- Did you find today's topic interesting?
- Was the this topic appropriately challenging relative to your background? That is, was the topic neither too elementary nor inaccessibly advanced?
- How could we improve this worksheet for future sessions?
- What did you enjoy about today's topic?
- What did you find particularly challenging?
- Was there anything you thought was too difficult?
- Was there anything you thought was too easy?
- Are there any topics you would be interested in seeing us cover in the future?

Pascal's Triangle

Abstract

Pascal's Triangle is a mathematical object that is simple to describe by a recursive definition, but that simplicity belies enormous richness and wide applicability of its structure. In this session, we shall introduce Pascal's Triangle, explore some of these properties, and present connections to branches of mathematics from combinatorics to geometry. This session is a continuation of our session from September 13, 2025.

Background needed: Prerequisites include basic algebra. Familiarity with *mathematical induction* will be helpful, especially to produce rigorous proofs for some of the exercises.

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Contents

0 Warmup

As prerequisites for this session, it will help to answer the following first. You do *not* need to know these answers already, and many questions will be revisited later in the worksheet.

> **Very Important:** Since Chapel Hill Math Circle meets only every two weeks during the academic year, one of my goals in writing this worksheet is to include plenty for you to explore between now and our next session. In particular, *I do not expect anyone—including the volunteers, and including myself!—to complete this worksheet in a mere 90 minute session.* This worksheet is intended to be more a marathon than a sprint. If you find yourself getting stuck on one problem or section, feel free to move to another.
>
> Conversely, this worksheet is hardly an exhaustive presentation of this topic, either. Pascal's Triangle has *so many properties* that it would be impossible, even in principle, to catalog even just those already discovered.

0.1 Let n be a nonnegative integer. What is $n!$, read as *n factorial*?

0.2 What is *mathematical induction*?

Note: Familiarity with mathematical induction will not be strictly necessary for this session, but it may be a useful method for providing rigorous proofs as solutions to some of the exercises below.

0.3 What are the *triangular numbers*? Can you explain why they called "triangular"?

1 Introducing Pascal's Triangle

1.1 Discussion

First, a bit of notation:

Notation. The notation

$$x := y \tag{1.1.1}$$

means that x is *defined to be equal to* y.

With that out of the way, let us introduce the core mathematical object for today's session:

Definition 1.1.1. *Pascal's Triangle*[1] is a triangular array that extends infinitely downward. Its first seven rows are given below:

row $n = 0$:							1							
row $n = 1$:						1		1						
row $n = 2$:					1		2		1					
row $n = 3$:				1		3		3		1				
row $n = 4$:			1		4		6		4		1			
row $n = 5$:		1		5		10		10		5		1		
row $n = 6$:	1		6		15		20		15		6		1	

Let $\binom{n}{k}$, read as "n choose k",[2] denote the kth entry, read left-to-right, in the nth row, where our numbering begins at the top with row $n = 0$ and on the left with $k = 0$. We set

$$\binom{0}{0} := 1, \tag{1.1.2}$$

[1] "Pascal's Triangle" is the most common name for this object in the English language. Recognition of the triangle predates Pascal by centuries, though, and it is also known as *meru-prastāra* (in India, meaning the Staircase of Mount Meru), the *Khayyam Triangle* (Iran), *Yang Hui's Triangle* (China), and *Tartaglia's Triangle* (Italy).

[2] These coefficients are typically defined in other ways that are conceptually different, but ultimately end up agreeing with how we are doing so relative to Pascal's Triangle.

and that for all integers $n \geq 0$,

$$\binom{n}{0} := 1, \text{ and } \binom{n}{n} := 1. \tag{1.1.3}$$

Further, $\binom{n}{k}$ is undefined if $k < 0$ or $k > n$. For all other pairs (n, k), we define $\binom{n}{k}$ inductively by *Pascal's Rule*:

$$\binom{n+1}{k} := \binom{n}{k} + \binom{n}{k-1}. \tag{1.1.4}$$

In other words, Pascal's Triangle has

- 1 at its top vertex,
- 1s at the outer edges of each row, and
- every other term is the sum of the two terms immediately above it.

As an example, the middle entry in row 6 is the sum of the two entries immediately above it, 10 (above and to the right) and 10 (above and to the left). Therefore, this entry is $10 + 10 = 20$.

Using the n choose k notation, the entries of first few rows of Pascal's Triangle are denoted as below:

$$\begin{array}{ccccccccc} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \end{array}$$

1.2 Exercises

1.2.1 Using the definition of Pascal's Triangle, what are the entries in row 7? What about row 8?

1.2.2 Note that in each row, the entries are identical whether we read right-to-left or left-to-right. Explain why every row is symmetric with respect to its center, not simply these early example rows.

1.2.3 What is the sum of all the entries in row 0? The sum over row 1, $1+1$? Row 2, where the sum is $1+2+1$? Row 3? If n is a nonnegative integer, what do you expect will be the sum of all the entries in row n?

Next consider the *alternating sums* of the entries in these first few rows, where we alternate between adding and subtracting consecutive terms: 1 for row 0, $1-1$ for row 1, $1-2+1$ in row 2, and $1-3+3-1$ for row 3. What are these respective sums? Can you provide a conjecture what the alternating sum is for the entries in row n in general?

1.2.4 Compute the values $11^0 = 1$, $11^1 = 11$, 11^2, 11^3, and 11^4. Compare these values to what we see in the entries through row 4. Does this continue when considering 11^5?

1.2.5 Consider the entries in the third diagonal of Pascal's Triangle: $\binom{2}{2} = 1, \binom{3}{2} = 3, \binom{4}{2} = 6, \binom{5}{2} = 10, \binom{6}{2} = 15, \binom{7}{2} = 21, \ldots$. Can you explain what this sequence represents?

1.2.6 Consider the following color-coded portion of Pascal's Triangle:

1
1 **1**
1 **2** 1
1 **3** 3 **1**
1 **4** 6 **4** 1
1 5 **10** **10** 5 1
1 6 15 20 **15** 6 1

(a) Verify that the sum of the blue numbers is equal to the red number: $1+2+3+4=10$.

(b) Verify that the sum of the orange numbers is equal to the green number: $1+4+10=15$.

(c) Without simply computing the sums in Exercises #1.2.6(a)–1.2.6(b), use Pascal's Rule (Equation (1.1.4)) to verify these equations.

(d) Let n, k be nonegative integers with $k \le n$. Prove the *Hockey-Stick Identity*:[3]

$$\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}. \tag{1.2.1}$$

In other words, if we take one of the 1s in the right hand diagonal—i.e., an entry of the form $\binom{k}{k}$, then for a diagonal beginning with that term—i.e., the terms $\binom{k}{k}$, $\binom{k+1}{k}$, $\binom{k+2}{k}$, …, $\binom{n}{k}$—the sum of these entries along the "shaft" of the hockey stick is equal to the single value $\binom{n+1}{k+1}$ in the "blade" of the stick.

In our examples in Exercises #1.2.6(a)–1.2.6(b), the first shaft is the blue sum $\binom{1}{1}+\binom{2}{1}+\binom{3}{1}+\binom{4}{1}=\mathbf{1}+\mathbf{2}+\mathbf{3}+4$, and the blade is the entry $\binom{5}{2}=\mathbf{10}$. The second shaft is formed by the orange entries, and the second blade is the green one.

[3]This gets its athletic name by locating each of the above elements in Pascal's Triangle. The resulting shape resembles a hockey stick.

Remark. Exercises #1.2.6(a) and 1.2.6(b) are, respectively, the special cases of the Hockey-Stick Identity for the cases $n = 4$, $k = 1$ and $n = 5$, $k = 3$, respectively. Note that by the symmetry of Pascal's Triangle, we can also have our diagonal sum arise from the leftmost diagonal rather than the rightmost one, providing a counterpart to Equation 1.2.1:

$$\binom{n-k}{0} + \binom{n-k+1}{1} + \binom{n-k+2}{2} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} \tag{1.2.2}$$

1.2.7 Consider the diagonals of Pascal's Triangle:

$$\begin{aligned}
\text{diagonal \#0} &: 1, 1, 1, 1, 1, 1, 1, \ldots \\
\text{diagonal \#1} &: 1, 2, 3, 4, 5, 6, 7, \ldots \\
\text{diagonal \#2} &: 1, 3, 6, 10, 15, 21, 28, \ldots \\
\text{diagonal \#3} &: 1, 4, 10, 20, 35, 56, 84, \ldots \\
\text{diagonal \#4} &: 1, 5, 15, 35, 70, 126, 210, \ldots
\end{aligned}$$

Using the Hockey-Stick Theorem, explain why diagonal #2 is the sequence of triangular numbers. (For context, recall Exercise 0.3.) Deduce that diagonal #3 is the sequence of *tetrahedral numbers*, where the nth tetrahedral number is defined to be the sum of the first n triangular numbers.

More generally, show that collection of these diagonal sequences give the *figurate numbers* of every order. If, for example, the triangular numbers (T_n) are the figurate numbers of order 2, with $T_n := 1 + 2 + \cdots + n$, then the tetrahedral numbers $(\mathcal{T}_n)$ are the figurate numbers or order 3, where $\mathcal{T}_n := T_1 + T_2 + \cdots + T_n$, the sum of the first n triangular numbers. In general, the nth figurate number of order k is the sum of the first n figurate numbers of order $k - 1$.

1.2.8 Let x, y be indeterminates. Verify that

$$(x + y)^2 = x^2 + 2xy + y^2. \tag{1.2.3}$$

Using (1.2.3), or some other way, expand $(x + y)^3$ and $(x + y)^4$. How do your answers compare to what we see in Definition 1.1.1 for Pascal's Triangle? Form a conjecture for how to compute $(x + y)^4$, $(x + y)^5$, and more generally, $(x + y)^n$ for every positive integer n.

1.2.9 In this exercise, we shall take sums in Pascal's Triangle along certain flattened diagonals (and with apologies to anyone with color blindness):

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & \mathbf{1} & & \mathbf{1} & & & & & \\
 & & & & \mathbf{1} & & 2 & & \mathbf{1} & & & & \\
 & & & 1 & & \mathbf{3} & & \mathbf{3} & & \mathbf{1} & & & \\
 & & \mathbf{1} & & \mathbf{4} & & \mathbf{6} & & 4 & & 1 & & \\
 & 1 & & \mathbf{5} & & 10 & & 10 & & 5 & & 1 & \\
 \mathbf{1} & & 1 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Then for these indicated flattened diagonals, we have

$$\binom{0}{0} = 1 = \boxed{1};$$

$$\binom{1}{0} = 1 = \boxed{1};$$

$$\binom{2}{0} + \binom{1}{1} = 1 + 1 = \boxed{2};$$

$$\binom{3}{0} + \binom{2}{1} = 1 + 2 = \boxed{3}.$$

Compute the next indicated sums for the purple, yellow, and red flattened diagonals.

Note that

green = blue + orange, and
pink = orange + green.

Prove that we also have

purple = pink + green,
yellow = purple + pink, and
red = yellow + purple.

Finally prove in general that beginning with flattened diagonal #3 (starting our count with #1), every sum is the sum of the previous two flattened diagonal sums.

Remark. Implicitly, I am asking you to prove that the sequence of sums over the flattened diagonals forms the *Fibonacci Sequence*. The Fibonacci Sequence (F_n) is defined with initial terms $F_1 := 1$ and $F_2 := 1$, with recurrence relation $F_n := F_{n-1} + F_{n-2}$ for all $n \geq 3$. Further, the Fibonacci Sequence has been a topic for Chapel Hill Math Circle's advanced group, most recently on both February 11 and February 25, 2023.

2 Computations and Identities in Pascal's Triangle

2.1 Discussion

In Section 1, we presented a definition of Pascal's Triangle, where for all $n \geq 0$, the entries in row $n+1$ are defined recursively in terms of those in row n. Our goal in this section is to produce a more direct way of computing the entries $\binom{n}{k}$ of Pascal's Triangle, without having to compute the entries $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$ beforehand. To begin, we must first provide an answer to Exercise #0.1 by defining n factorial:

Definition 2.1.1. Let n be a nonnegative integer. Then n *factorial*, denoted $n!$, is defined as

$$n! := \begin{cases} 1, & \text{if } n = 0 \\ 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n, & \text{otherwise.} \end{cases} \tag{2.1.1}$$

That is, when n is a positive integer, $n!$ is the product of the first n positive integers, and $0!$ is defined to be 1 as a special case. The first few factorials are $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, and $6! = 720$. Note that the sequence of factorial grows very rapidly!

Having defined $n!$, our goal for this section is to prove the following identity, after which we shall find applications for this formula:

Theorem 2.1.2. *Let n be a nonnegative integer. If k is an integer such that $0 \leq k \leq n$, then the kth entry (starting from $k = 0$) in row n (starting from $n = 0$) in Pascal's Triangle, $\binom{n}{k}$, is given by the formula*

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}. \tag{2.1.2}$$

Example 2.1.3. From row 6 of Pascal's Triangle as given in Definition 1.1.1, we have that $\binom{6}{2} = 15$. Furthermore, we have

$$\begin{aligned}\frac{6!}{2!\cdot(6-2)!} &= \frac{6!}{2!\cdot 4!}\\ &= \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6}{(1\cdot2)\cdot(1\cdot2\cdot3\cdot4)}\\ &= \frac{5\cdot6}{1\cdot2}, \text{ cancelling the common factor } 4! = 1\cdot2\cdot3\cdot4\\ &= \frac{30}{2}\\ &= 15,\end{aligned}$$

verifying that this formula is correct for this particular entry.

Remark. Note that in the computations for Example 2.1.3, we didn't need to compute either 6! or 4! directly. This method of cancelling lots of common factors in the numerator and denominator of $\frac{n!}{k!\cdot(n-k)!}$ will be extremely helpful for computing the right-hand side of (2.1.2) for given values of n and k.

2.2 Exercises

2.2.1 Verify Equation (2.1.2) holds for $n = 4$, $k = 2$

2.2.2 Verify Equation (2.1.2) holds for $n = 6$, $k = 3$

2.2.3 In Definition 1.1.1, we began with the initial conditions $\binom{0}{0} := 1$, and for every positive integer n, $\binom{n}{0} = 1$, and $\binom{n}{n} = 1$. Prove that for every nonnegative integer n,

$$\frac{n!}{0!\cdot(n-0)!} = 1, \text{ and} \tag{2.2.1}$$

$$\frac{n!}{n!\cdot(n-n)!} = 1.. \tag{2.2.2}$$

In other words, for every n, Equation 2.1.2 holds for the specific cases $k = 0$ and $k = n$.

2.2.4 Let n, k be nonnegative integers with $k \le n-1$. Prove that

$$\frac{(n+1)!}{k! \cdot [(n+1)-k]!} = \frac{n!}{k! \cdot (n-k)!} + \frac{n!}{(k-1)! \cdot [n-(k-1)]!}. \tag{2.2.3}$$

That is, verify that our formula $\frac{n!}{k!\cdot(n-k)!}$ also satisfies satisfies Pascal's Rule, Equation (1.1.4), in the sense that the "bottom" term $\frac{(n+1)!}{k!\cdot[(n+1)-k]!}$ is the sum of the two terms immediately "above" it.

2.2.5 Using the result of Exercises #2.2.3–2.2.4, prove that for all nonnegative integers n, k with $k \le n$,

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}. \tag{2.2.4}$$

2.2.6 Compute $\binom{12}{5}$.

2.2.7 In diagonal #2, the diagonal of triangular numbers, prove that the sum of two adjacent terms is a perfect square. For example, we have $1+3=4=2^2$, $3+6=9=3^2$, $6+10=16=4^2$, $10+15=25=5^2$, etc.

2.2.8 Choose an entry in Pascal's Triangle not lying on the edge. This term will be surrounded by six adjacent neighbors forming a hexagon. For example, in row 4, take the entry $\binom{4}{2}=6$. This is surrounded by the neighboring entries $\binom{3}{1}=3$ (above and left), $\binom{3}{2}=3$ (above and to right), $\binom{4}{3}=4$ (immediate right), $\binom{5}{3}=10$ (below and right), $\binom{5}{2}=10$ (below and left), and $\binom{4}{1}=4$ (immediate left).

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & \mathbf{3} & & \mathbf{3} & & 1 & & & \\
 & & 1 & & \mathbf{4} & & \mathbf{6} & & \mathbf{4} & & 1 & & \\
 & 1 & & 5 & & \mathbf{10} & & \mathbf{10} & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Starting with $\binom{3}{1}$, take every other vertex on the hexagon, and take the product of these blue entrie=s: $3 \cdot 4 \cdot 10 = 120$. Similarly, take the remaining red entries: $3 \cdot 10 \cdot 4 = 120$. In this case, these products are equal.

Prove that this identity holds in general: for an entry in the interior of the triangle, the respective products of the blue entries and of the red entries are always equal.

2.2.9 **Potentially Challenging:**[4] Let P_n denote the product of the terms in row n of Pas-

[4] I label this potentially challenging because this problem requires familiarity with the number $e = 2.71828...$, which is typically defined as the *limit* of a particular sequence: $e := \lim_{n\to\infty}(1+1/n)^n$. If you have not yet studied this number or you have not worked with limits, feel free to skip this exercise.

cal's Triangle:

$$P_n := \binom{n}{0} \cdot \binom{n}{1} \cdots \binom{n}{n}. \tag{2.2.5}$$

Prove that

$$\frac{P_{n+1} \cdot P_{n-1}}{P_n^2} = \left(1 + \frac{1}{n}\right)^n. \tag{2.2.6}$$

Conclude that

$$\lim_{n\to\infty} \frac{P_{n+1} \cdot P_{n-1}}{P_n^2} = e. \tag{2.2.7}$$

Note: This exercise is taken from [3].

3 Pascal's Triangle and Combinatorics

3.1 Discussion

One important application of Pascal's Triangle is that its entries count certain fundamental quantities in *combinatorics*, a branch of math focused on counting numbers of certain objects. In this section, we shall explore some of the combinatorial properties of Pascal's Triangle. Much of this section has been adapted from Section 1 of "Thinking Combinatorially", the worksheet for the advanced group of Chapel Hill Math Circle on our session of March 19, 2022.

To begin, let us use the following convention to simplify notation:

Notation. Let n be a positive integer. Then we set

$$[\mathbf{n}] := \{1, 2, \ldots, n\}. \tag{3.1.1}$$

Further, if $n = 0$, then we define $[\mathbf{n}] := \varnothing$, the *empty set*, which is the unique set containing no elements.

Definition 3.1.1. Let n be a nonnegative integer, and let k be any integer. We define $C(n,k)$ to be the number of distinct k-element sets lying in $[\mathbf{n}]$.

Example 3.1.2. Consider the case $n := 3$, $k := 2$. By inspection, we see that the distinct 2-element subsets of $[\mathbf{2}] = \{1,2,3\}$ are precisely $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$; every other subset of $[\mathbf{3}]$ is either the wrong size, or it's already on our list. Since there are 3 such 2-element subsets of $[\mathbf{3}]$, it follows that $\boxed{C(3,2) = 3}$.

Example 3.1.3. Consider the case $n := 4$, $k := 2$. By inspection, we see that the distinct 2-element subsets of $[\mathbf{4}] = \{1,2,3,4\}$ are precisely $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, and $\{3,4\}$. Since there are 6 such 2-element subsets, it follows that $\boxed{C(4,2) = 6}$.

Example 3.1.4. Consider the case $n := 4$, $k := 5$. Then by definition, $C(4,5)$ counts the number of distinct 5-element subsets of $[\mathbf{4}]$, a set with precisely four elements. There are no such subsets of $[\mathbf{4}]$ containing five elements, though, so $\boxed{C(4,5) = 0}$. (Indeed, for this same reason, in general we have that $C(n,k) = 0$ whenever $k > n$ or $k < 0$.)

3.2 Exercises

3.2.1 Compute $C(5,2)$, which counts the number of 2-element subsets of $[\mathbf{5}]$.

Hint: Note that our computations in Example 3.1.3 have already enumerated several 2-element subsets of $[\mathbf{5}]$: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, and $\{3,4\}$. How can you find the other distinct 2-element subsets of $[\mathbf{5}]$ not already our list for computing $C(4,2)$? In particular, how many such subsets are there?

3.2.2 Let n be a positive integer and k any integer. Prove that

$$C(n,k) = C(n,n-k). \tag{3.2.1}$$

That is, the number of k-element subsets of $[\mathbf{n}]$ is the same as the number of $(n-k)$-element subsets of $[\mathbf{n}]$.

3.2.3 If n, k are integers, then

$$C(n+1,k) = C(n,k) + C(n,k-1). \tag{3.2.2}$$

In other words, $C(n,k)$ also satisfies its own version of Pascal's Rule (Equation (1.1.4)).

3.2.4 Let n be any nonnegative integer. Prove that $C(n,0) = C(n,n) = 1$. Explain how we can therefore conclude that for all integers k with $0 \le k \le n$,

$$C(n,k) = \binom{n}{k}. \tag{3.2.3}$$

That is, the number of k-element subsets of an n-element set is precisely the entry $\binom{n}{k}$ in Pascal's Triangle.

3.2.5 Let n, k be nonnegative integers, with $k \le n$. Consider an $(n+1) \times (k+1)$ grid, with outermost points $(0,0)$ and $(n+1, k+1)$ (shown In Figure 3.2.1 for the specific case $n := 8$, $k := 5$: Starting from the origin, we seek to count the number of paths from

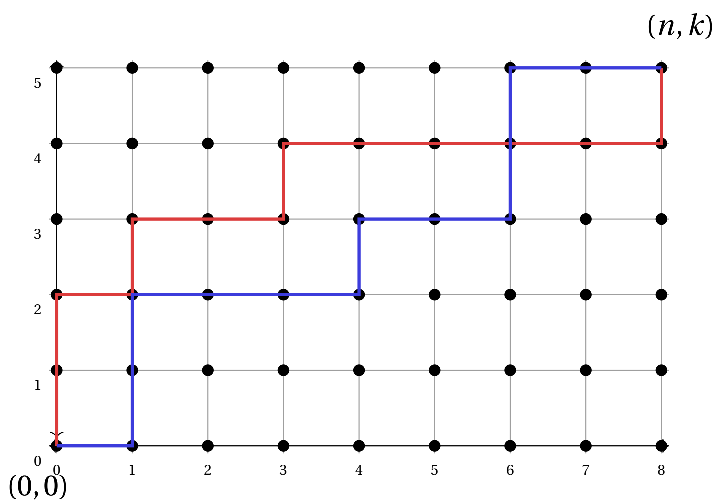

Figure 3.2.1: Two paths from $(0,0)$ to (n,k).

the origin $(0,0)$ to the point (n,k), where our only legal moves within a path are to move up one step or right one step, never going backwards, retracing, or jumping over adjacent nodes. In the grid above, the blue and red paths are examples of valid paths.

(a) How many distinct valid paths are there from $(0,0)$ to $(1,0)$? From $(0,0)$ to $(0,1)$? From $(0,0)$ to $(1,1)$? To $(2,0)$, $(2,1)$, and $(2,2)$, respectively?

(b) By counting, explain how there are precisely $C(n+k,k)$ distinct valid paths from $(0,0)$ to (n,k) under these rules.

Hint: How many total "rightward" moves are in a valid path from the origin to (n,k)? How many "upward" moves, and how many total moves?

(c) Show that the number of paths from $(0,0)$ to (n,k) is equal to the number of paths from $(0,0)$ to $(n-1,k)$ *plus* the number of paths from $(0,0)$ to $(n-1,k-1)$. In other words, our path-counting procedure again follows something analogous to Pascal's Rule.

4 The Binomial Theorem

4.1 Discussion

In Exercise #1.2.8, we expanded $(x+y)^2$, $(x+y)^3$, and saw that the coefficients for these expansions were given, respectively, by the terms in rows 2 and 3 of Pascal's Triangle. This suggests the following generalization, which we shall now consider in more detail:

Theorem 4.1.1 (The Binomial Theorem)**.** *Let x, y be indeterminates, and let n be a positive integer. Then*

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n. \tag{4.1.1}$$

That is, expanding $(x+y)^n$, then collect like terms. Then coefficient of the term $x^k y^{n-k}$ is precisely the jth term (starting with $k=0$), counting from the left, in row n of Pascal's Triangle, or $\binom{n}{k}$.

4.2 Exercises

4.2.1 *Provided* you are familiar with mathematical induction—see Exercise #0.2—prove the Binomial Theorem via induction.

4.2.2 Using our identity $\binom{n}{k} = C(n,k)$ for $0 \le k \le n$ in Equation (3.2.3), prove the Binomial Theorem using combinatorics.

Hint: Hints: In expanding $(x+y)^n = \underbrace{(x+y)\cdots(x+y)\cdot\cdots\cdot(x+y)}_{n \text{ times}}$, consider the coefficient of $x^j y^{n-j}$. How does this term arise from the products, and how can we express this combinatorially?

4.2.3 Assuming the Binomial Theorem is true (whether or not you have proven it yet), prove that the sum of all the entries in row n of Pascal's Triangle is 2^n. (Compare to Exercise #1.2.3.)

4.2.4 Again, assuming the Binomial Theorem is true, prove that the alternating sum of all the entries in row n of Pascal's Triangle is either 1 when $n = 0$, and the sum is 0 otherwise.

That is, prove that when $n \ge 1$,

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0. \tag{4.2.1}$$

Conclude that for all $n \geq 1$, there are precisely as many subsets of odd size as there are subsets of even size.

Hint: For the odd/even question, note that Equation (4.2.1) holds if and only if

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots \tag{4.2.2}$$

(Each sum above is ultimately of finitely many terms, but an expression for the final term in each sum would be tedious and unilluminating.)

4.2.5 In Exercise #1.2.4, we noted in early rows of Pascal's Triangle, concatenating the digits forms powers of 11: $11^1 = 11$, $11^2 = 121$, $11^3 = 1331$, and $11^4 = 14641$. Using the Binomial Theorem, explain how we can deduce this without having to manually compute these powers of 11. Why does our pattern break down when $n \geq 5$?

5 Number Theory and Fractal Patterns in Pascal's Triangle

5.1 Discussion

One important technique in number theory is that *modular arithmetic*: when reducing the integers modulo a positive integer $m > 1$, we are interested in the *remainders* when dividing by m.

For an example, take Pascal's Triangle, and reduce it modulo 2. That is, replace any odd term in Pascal's Triangle with 1, and replace any even term with 0:

$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&1&&1&&&&&& \\
&&&&&1&&0&&1&&&&& \\
&&&&1&&1&&1&&1&&&& \\
&&&1&&0&&0&&0&&1&&& \\
&&1&&1&&0&&0&&1&&1&& \\
&1&&0&&1&&0&&1&&0&&1& \\
1&&1&&1&&1&&1&&1&&1&&1
\end{array}$$

Note, in particular, that we can easily compute Pascal's Triangle modulo 2 *without* computing the original entries in Pascal's Triangle itself. This is because on the interior of Pascal's Triangle, so long as we know which of $\binom{n}{k-1}$ and $\binom{n}{k}$ is odd and which is even, we know that even + even = even, odd + odd = even, and even + odd = odd + even = odd. We express this using modular arithmetic by saying $0+0 \equiv 0 \pmod 2$, $1+1 \equiv 0 \pmod 2$, and $0+1 \equiv 1+0 \equiv 1 \pmod 2$.

Separately, some of you may already be familiar with *Sierpiński's Triangle*,[5] a canonical example of a fractal, approximated by Figure 5.1.1:

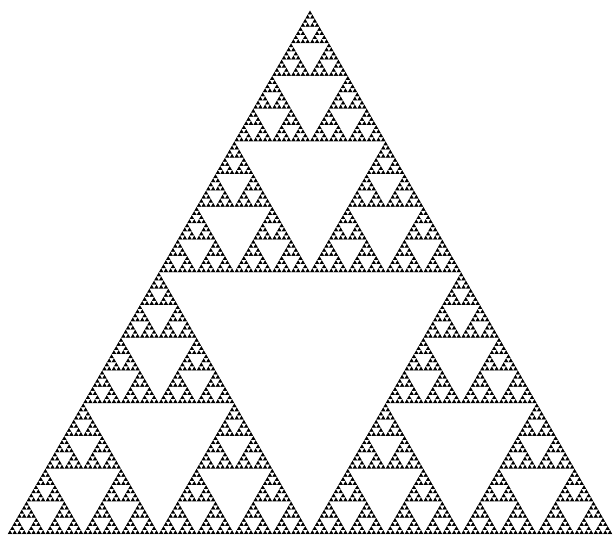

Figure 5.1.1: An approximation of the fractal Sierpiński's Triangle.

The rule to form Sierpiśski's Triangle is indicated in Figure 5.1.2: begin with a solid, black equilateral triangle, remove its middle fourth, then successively iterate this process of removing middle fourths from any remaining solid equilateral triangle.

[5]The namesake for Sierpiński's Triangle is Polish mathematician *Wacław Sierpiński* (1882–1969). (Incidentally, according to both Wikipedia and Google Translate, his name is roughly pronounced "Vahts-wahf CHER-peen-skee".)

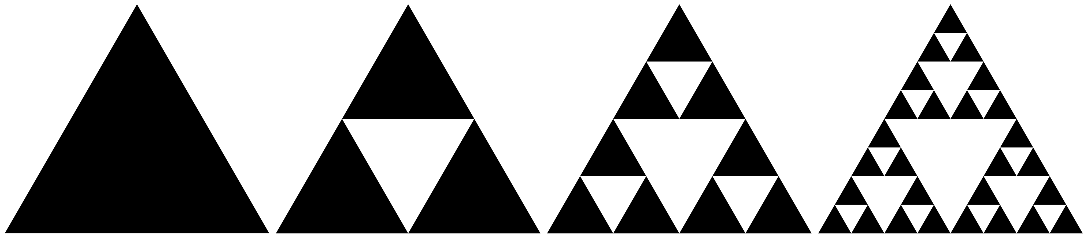

Figure 5.1.2: The iterative step for forming Sierpiński's Triangle. The true fractal is the limit (in a sense that can be made precise) of iterating this rule of replacing the middle fourth of a solid equilateral triangle

5.2 Exercises

5.2.1 Expand the above to compute Pascal's Triangle mod 2 through row 15. which rows have all 1s? Which have all 0s other than at the endpoints?

5.2.2 **Open Question:** What patterns do you notice? Can you form any conjectures, and if so, can you prove them?

5.2.3 **Open Question:** Repeat the above procedure, reducing Pascal's Triangle modulo m for various positive integers $m > 2$. Do you notice any patterns? Can you form any conjectures, and if so, can you prove them?

5.2.4 In Pascal's Triangle, row 2 is 1 2 1, row 3 is 1 3 3 1, row 5 is 1 5 10 10 5 1, and row 7 is 1 7 21 35 35 21 1. Each of these row indices is prime, and in each row, other than the 1s on the endpoints, every term is divisible by the row number

Prove this is true in general: if p is a positive prime, then the terms

$$\binom{p}{1}, \binom{p}{2}, \dots, \binom{p}{p-1}$$

are all divisble by p.

Remark. Note that a similar result does *not* hold when the row number is not prime. For example, in row 4, $\binom{4}{2} = 6$, and 6 is not a multiple of 4.

5.2.5 **Challenging:** Let p be a prime, and consider the pth right-to-left diagonal

$$\binom{p-1}{p-1}, \binom{p}{p-1}, \binom{p+1}{p-1}, \binom{p+2}{p-1}, \dots$$

(Or, if you prefer, consider its left-to-right reflection

$$\binom{p-1}{0}, \binom{p}{1}, \binom{p+1}{2}, \binom{p+2}{3}, \dots,$$

instead). Prove that

$$\binom{n}{p-1} \text{ is a multiple of } p \qquad \text{if and only if} \qquad n \not\equiv -1 \pmod{p}. \tag{5.2.1}$$

Example 5.2.1. I'm sure a few examples would help explain what's going on:

- The diagonal for the prime $p = 3$ is $\binom{2}{2} = 1$, $\binom{3}{2} = 3$, $\binom{4}{2} = 6$, $\binom{5}{2} = 10$, $\binom{6}{2} = 15$, and so on. Of these every third entry is not divisible by 3—namely 1, 10, etc.—but all other terms are divisble by 3.
- The diagonal for the prime $p = 5$ is $\binom{4}{4} = 1$, $\binom{5}{4} = 5$, $\binom{6}{4} = 15$, $\binom{7}{4} = 35$, $\binom{8}{4} = 70$, $\binom{9}{4} = 126$, $\binom{10}{4} = 210$, and so on. Of these, all but 1 and 126 are divisible by 5.
- The diagonal for the prime $p = 7$ passes through $\binom{6}{6} = 1$, $\binom{7}{6} = 7$, $\binom{8}{6} = 28$, $\binom{9}{6} = 84$, $\binom{10}{6} = 210$, $\binom{11}{6} = 462$, $\binom{12}{6} = 924$, $\binom{13}{6} = 1716$, $\binom{14}{6} = 2002$, and so on. Of these, 1 and 1716 are not divisble by 7, and all these other terms are.

The claim is that for *every* prime p, every pth term on the diagonal, starting with the rightmost $1 = \binom{p-1}{p-1}$ is *not* divisible by p, and every other term is.

Note: This appears in [25].

5.2.6 **Challenging:** As in Exercise #5.2.1, reduce Pascal's Triangle modulo 2, so that odd entries are replaced by 1 and even entries replaced by 0. Let S_n denote the sum of the resulting entries in row n as an integer *without further reducing mod* 2. What can you say about S_n?

Example 5.2.2. A few examples might be helpful to illustrate the underlying idea here.

- In the original Pascal's Triangle, row 0 is simply 1. Reducing mod 2, we obtain 1 again, and the sum of these reduced entries is $\boxed{1}$.
- In the original Pascal's Triangle, row 1 is 1 1. Reducing mod 2, we obtain 1 1 again, and the sum of these reduced entries is $\boxed{2}$.
- In the original Pascal's Triangle, row 2 is 1 2 1. Reducing mod 2, we obtain 1 0 1, and the sum of these reduced entries is $\boxed{2}$.
- In the original Pascal's Triangle, row 3 is 1 3 3 1. Reducing mod 2, we obtain 1 1 1 1, and the sum of these reduced entries is $\boxed{4}$.
- In the original Pascal's Triangle, row 4 is 1 4 6 4 1. Reducing mod 2, we obtain 1 0 0 0 1, and the sum of these reduced entries is $\boxed{2}$.
- In the original Pascal's Triangle, row 5 is 1 5 10 10 5 1. Reducing mod 2, we obtain 1 1 0 0 1 1, and the sum of these reduced entries is $\boxed{4}$.
- In the original Pascal's Triangle, row 6 is 1 6 15 20 15 6 1. Reducing mod 2, we obtain 1 0 1 0 1 0 1, and the sum of these reduced entries is $\boxed{4}$.

As a preliminary matter to solving this exercise, one might make the following observations:

- Among these examples, for all $n \geq 1$, S_n is even.
- Among these examples, for all n, S_n is a nonnegative integer power of 2.

- Among these examples, for all $n \geq 1$, $S_{2n} = S_n$. For example, $S_2 = S_1 = 2$, $S_4 = S_2 = 2$, $S_6 = S_3$, etc.
- Among these examples, for all $n \geq 0$, $S_{2n+1} = 2S_n$. For example, $S_1 = 2S_0$, $S_3 = 2S_1$, $S_5 = 2S_2$, etc.

Which of these patterns continue? Can you find any others, and can you prove them? Finally, what is the strongest conjecture you can make regarding how S_n behaves as a function of n, and can you prove it?

Note: This problem, taken from [6] in [7], is equivalent to Power Question II(c) from the 1986 *New York State Mathematics League (NYSML)*.

Hints and Partial Answers: Hints might depend on what the kids find on their own. To see that S_n is even for $n \geq 1$, note that by Exercises #prob:sums and alternating sums of the entries in row n and 4.2.3, the *actual* sum over row n, without reducing mod 2, is 2^n. Reducing mod 2 is therefore equivalent to reducing 2^n itself mod 2, and this is always 0 for all $n \geq 1$. For an alternate method, recall the symmetry of Pascal's Triangle: for all nonnegative integers n, k with $0 \leq k \leq n$, $\binom{n}{k} = \binom{n}{n-k}$, whence $\binom{n}{k} \equiv \binom{n}{n-k} \pmod 2$. It follows that the pair $\binom{n}{k}, \binom{n}{n-k}$ sums to 0 mod 2, provided these elements are distinct. Can you work the edge cases to see what happens when $k = n - k$? (Note that $n = 0$ is an exceptional case to this exception.)

Regarding the question of the strongest conjecture one might have for S_n, according to [6], one of the strongest conjectures one might form and prove is the following: for all n, $S_n = 2^{B(n)}$, where $B(n)$ is the sum of the digits in the binary representation of n.

For example, if $n = 3$, then $n = 11_2$. The sum of these binary digits is 2, and $2^{B(3)} = 2^2 = 4 = S_3$, as claimed. Similarly, if $n = 5$, then $n = 101_2$, and the sum of these binary digits is again 2. Then $2^{B(5)} = 2^2 = 4 = S_5$, again as claimed.

As for proving this claim, it would suffice to prove the conjectures about how $S_{2n} = S_n$ and $S_{2n+1} = 2S_n$. To see why, think about the binary representations of $2n$ and $2n + 1$ in terms of that of n. If

$$n = (a_m a_{m-1} \cdots a_1 a_0)_2$$

is the binary representation of n, then $2n$ and $2n + 1$ have binary representations

$$\begin{aligned} 2n &= (a_m a_{m-1} \cdots a_1 a_0 0)_2 \\ 2n + 1 &= (a_m a_{m-1} \cdots a_1 a_0 1)_2 . \end{aligned}$$

It follows that $S(2n) = S(n)$ and $S(2n+1) = S(n)+1$, so some variant of induction can complete the argument. Of course, one would still have to prove the conjectures, but that's why this exercise is labeled as challenging! □

6 Pascal's Triangle and Probability

6.1 Discussion

Plinko is a game from the TV game show *The Price Is Right*. Consider the following simplification of Plinko, with a game board like what is shown in Figure 6.1.1: Assume that,

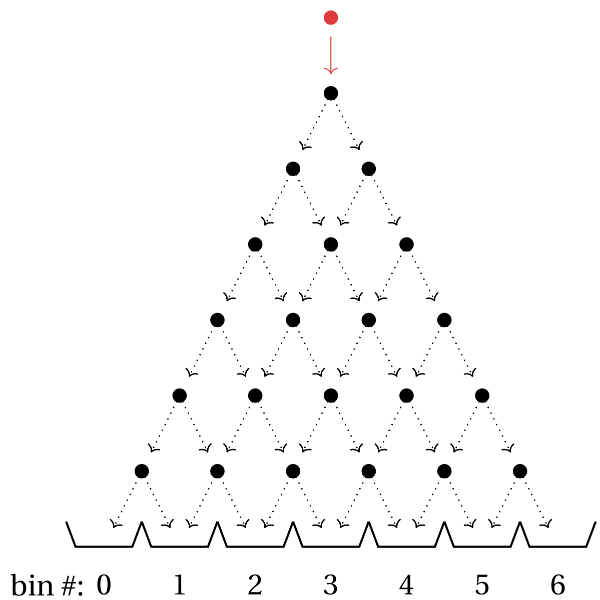

Figure 6.1.1: Plinko-like game, where we drop the red ball from the indicated position. At each collision with a black point, the red ball has probability p_{right} of bouncing down and to the right, and probability $p_{\text{left}} = 1 - p_{\text{right}}$ of bouncing down and to the left.

in Figure 6.1.1, we have that at each black point, the red point has probability $p_{\text{right}} = \frac{1}{2}$ of bouncing down and to the right, and probability $p_{\text{left}} = 1 - p_{\text{right}} = \frac{1}{2}$ of bouncing down and to the left. Further, assume that the ball stops in one of the numbered bottom bins. We are interested in the overall probability of the red ball ultimately falling in bin $\#k$.

In the exercises below, we shall consider the probabilities of our red ball landing in each of these bins, then consider additional properties and generalizations of this probability distribution.

6.2 Exercises

6.2.1 Consider a game where you take a fair coin, so that $p_{\text{heads}} = p_{\text{tails}} = \frac{1}{2}$, and you flip it six times in succession, then count the total number of heads over all six coin flips. Explain why this game is equivalent to the Plinko variant described in Section 6.1 above.

6.2.2 Clearly we have precisely 7 distinct outcomes, where the ball lands in bin #0, bin #1, and so on up to bin #6. How many total paths are there from the initial position of the red ball to the bins below?

6.2.3 How many paths are possibly for the red ball to end up in bin #0, and what is the probability of the red ball ending up in bin #0? Repeat these question for bin #6.

6.2.4 For each k with $0 \leq k \leq 6$, how many paths in general are there from the top to bin #k in row #6? What is the probability of ending in bin #k? Use this to compute how many paths there are to bin #2, then compute the probability of ending in bin #2.

Note: Compare this result to that of Exercise #3.2.5.

6.2.5 Generalize our game of Plinko so that the ball terminates in row #n, in one of $n+1$ bins labeled #0, #1, ..., #n.

6.2.6 Throughtout Exercises #6.2.2–6.2.5, we assumed that $p_{\text{right}} = \frac{1}{2}$ of bouncing down and to the right, and probability $p_{\text{left}} = 1 - p_{\text{right}} = \frac{1}{2}$ of bouncing down and to the left.

Assume, instead, that the pair of p_{right} and p_{left} is identical at each black point, but that we need not have $p_{\text{left}} = p_{\text{right}} = \frac{1}{2}$. Express the probability that the ball ends in bin #k in terms of $p := p_{\text{right}}$.

6.2.7 In Exercise #6.2.6, one can show that the probability of terminating in bin #k of row #n is

$$\binom{n}{k} p^k (1-p)^{n-k}, \tag{6.2.1}$$

where $p = p_{\text{right}}$; in general, a probability distribution satisfying something analogous to Equation (6.2.1) is called a *binomial distribution*.

One defining property required by a valid probability distribution is that the probability of *some* event must be 1, or 100%. Prove that for a binomial distribution, this is true: adding over all valid k, we have that for all

$$\binom{n}{0} p^0 (1-p)^n + \binom{n}{1} p^1 (1-p)^{n-1} + \cdots + \binom{n}{n} p^n (1-p)^0 = 1. \tag{6.2.2}$$

6.2.8 In our first session this semester on August 30, 2025, our topic was *expected value*. In general, expected value is a probability-weighted measurement of what we expect, on average, to be the result of repeating various trials. For example, if we have a fair 6-sided dice and roll it once, the expected value is the sum over multiplying the probability of rolling a k and k itself:

$$\mathbb{E} = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6 = \boxed{\frac{7}{2}}. \tag{6.2.3}$$

Consider a binomial distribution satisfying Equation (6.2.6) with probabilities p (for success) and $1-p$ (for failure) with n trials. If X is such a hrefhttps://en.wikipedia.org/wiki/Rando *variable*, one can express this with the notation $X \sim B(n,p)$, meaning "X is distributed as (or follows, or any of a number of other equivalent terms) the binomial distribution with parameters n and p". Letting $\mathbb{E}[X]$ denote the expected value of X, prove that

$$\mathbb{E}[X] = np. \tag{6.2.4}$$

Hint: To prove (6.2.4) using the formula in Equation (6.2.1), it may help to first prove the following: for all positive integers n, k with $1 \le k \le n$,

$$k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}. \tag{6.2.5}$$

Using this, find a common factor of np in Equation (6.2.1), then simplify the remaining sum.

7 Variants and Generalizations of Pascal's Triangle

7.1 Discussion

Finally, we briefly introduce some generalizations of Pascal's Triangle. First, consider the *Leibniz Harmonic Triangle*: here, we again begin with 1 at the top vertex. Now, at the edges of row n, we have the values $\frac{1}{n}$. And finally, every term is the sum of the two elements *below* it:

$$\begin{array}{lccccccccccccc}
\text{row } n=0: & & & & & & & 1 & & & & & & \\
\text{row } n=1: & & & & & & \frac{1}{2} & & \frac{1}{2} & & & & & \\
\text{row } n=2: & & & & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & & & \\
\text{row } n=3: & & & & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & & & \\
\text{row } n=4: & & & \frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} & & \\
\text{row } n=5: & & \frac{1}{6} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{60} & & \frac{1}{30} & & \frac{1}{6} & \\
\text{row } n=6: & \frac{1}{7} & & \frac{1}{42} & & \frac{1}{105} & & \frac{1}{140} & & \frac{1}{105} & & \frac{1}{42} & & \frac{1}{7}
\end{array}$$

So, for example, in row 6, the (left) term $\frac{1}{60}$ arises as the sum $\frac{1}{60} = \frac{1}{105} + \frac{1}{140}$ of the terms immediately below it.

Next, we consider a three-dimensional extension of Pascal's Triangle: *Pascal's Pyramid* (or *Pascal's Tetrahedron*):

Here, rather than having one-dimensional rows as the relevant slices, we have two-dimensional triangular arrays at each level (other than at the very top for $n = 0$). Further,

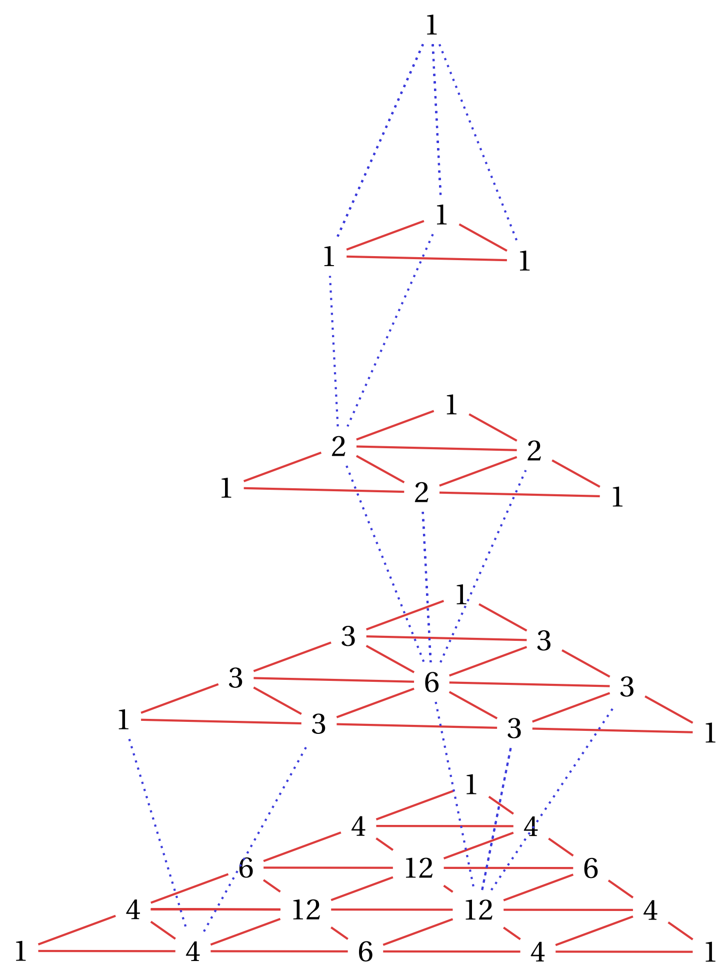

Figure 7.1.1: Pascal's Pyramid for levels 0 through 4.

for points in the interior, we compute them by adding the *three* terms above, rather than the two upstairs neighbors for Pascal's Triangle.

In Pascal's Triangle, the "coordinates" for the entry $\binom{n}{k}$ are straightforward enough: this lies on row n, and is the kth entry, counting from the left. In other words, n tells us which one-dimensional slice we're considering in the triangle, and k gives us the position within row n. In Pascal's Pyramid, to determine a position for an entry, we need the level of the two-dimensional slice in terms of its vertical level, n. Then within slice n, we will need to to specify its two-dimensional position there. A useful way to do this is using a *barycentric coordinate system*, where within slice n, we specify relative closeness to each of the outer triangular vertices of that slice. Similarly, we will need to generalize our notation for binomial coefficients, $\binom{n}{k}$, when ascending to higher dimensions. In Pascal's

Pyramid, entries will be denoted in the form

$$\binom{n}{k_1, k_2, k_3}, \text{ where } k_1, k_2, k_3 \text{ are nonnegative integers such that } k_1 + k_2 + k_3 = n. \quad (7.1.1)$$

These entries are related to the coefficients of the monomials $x^{k_1} y^{k_2} x^{k_3}$ in expanding $(x+y+z)^n$, and this connection between Pascal's Pyramid and a *Trinomial Theorem* can also provide an independent motivation for how we provide coordinates for terms in the Pyramid.

As mentioned above, in general, a term in Pascal's Pyramid will be the sum of the three entries immediately above it. Identifying exactly which terms are *immediately* above $\binom{n}{k_1,k_2,k_3}$, though, is harder to picture three-dimensionally. Algebraically, we have the following counterpart to Pascal's Rule, Equation (1.1.4): if n is a nonnegative integer, and k_1, k_2, k_3 are *positive* integers such that $k_1 + k_2 + k_3 = n + 1$, then

$$\binom{n+1}{k_1, k_2, k_3} = \binom{n}{k_1 - 1, k_2, k_3} + \binom{n}{k_1, k_2 - 1, k_3} + \binom{n}{k_1, k_2, k_3 - 1}. \quad (7.1.2)$$

Here, $n+1$ is the level of the next level, and the terms $\binom{n}{k_1-1,k_2,k_3}$, $\binom{n}{k_1,k_2-1,k_3}$, and $\binom{n}{k_1,k_2,k_3-1}$ are the terms in level n immediately above $\binom{n+1}{k_1,k_2,k_3}$. In particular, note that if $k_1+k_2+k_3 = n+1$. then $(k_1-1)+k_2+k_3 = k_1+(k_2-1)+k_3+k_1+k_2+(k_3-1) = n$, which we need for these immediate upstairs neighbors to lie in level n in the first place.

Of course, there's no need to limit ourselves to a three-dimensional generalization like Pascal's Pyramid: though we as humans can't really picture four- or higher-dimensional extensions, these definitely exist as perfectly valid mathematical objects. In a four-dimensional generalization of Pascal's Triangle, for example, level n will now be a three-dimensional object, so that level n is itself a (finite) tetrahedron. Going up to dimension 5, level n will be a four-dimensional object; in general, level n of a d-dimensional generalization of Pascal's Triangle will have dimension $d-1$.

7.2 Exercises

7.2.1 **Open Question:** Can you see any patterns in the Leibniz Harmonic Triangle? Can you express the terms this relative to those of Pascal's Triangle?

7.2.2 **Open Question:** What properties do you notice in Pascal's Pyramid? For example, can you compute the entries in other levels? Why do we have copies of Pascal's

Triangle on the outer faces of the tetrahedron? Do we have any generalization of the Binomial Theorem to Pascal's Tetrahedron? What are the sums of the elements on each level? What is a closed-form formula for terms of the pyramid?

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