Alien Arithmetic

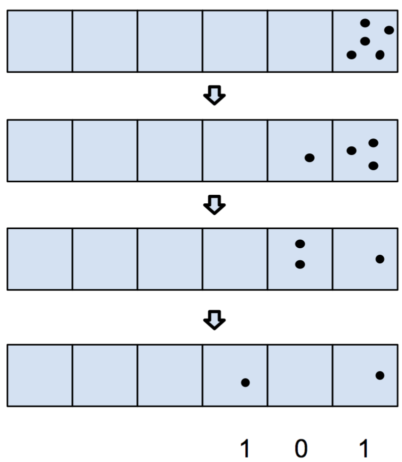
April 26, 2025

1 James Tanton's Exploding Dots

A $1 \leftarrow 2$ machine consists of a row of boxes, extending to the left as far as you'd like. To operate the machine, place a number of dots in the right most box. The machine then redistributes the dots according to the rule:

> *Two dots in any one box vanish (they explode) and are replaced with one dot one box to their left.*

When all the explosions have died down, you can read off a code of 1's and 0's representing the number of dots in each box.



1. What happens if you start with six dots? Eight dots? 25 dots?
2. Now use a $1 \leftarrow 6$ machine. Encode 19. Now encode 42.
3. If your friend came up with the code 152 using a $1 \leftarrow 6$ machine, how many dots did she start with?
4. What is the code for 253 using a $1 \leftarrow 10$ machine?

2 Number Bases

From now on, a number written with a subscript will mean that that number is written using a the subscript as a base. That is, 11_6 means 11 base 6 – that is, the number encoded as 11 in a $1 \leftarrow 6$ machine, which is the number 7. A number without a base means ordinary base 10. So, for example, $20_6 = 12$ and $113_6 = 45$.

5. Write these numbers in base 10:
   (a) 15_7
   (b) 35_7
   (c) 45_7
   (d) 412_7
6. Write these numbers in base 7:
   (a) 13
   (b) 48
   (c) 63
   (d) 625
   (e) 1000

Extra problems:

7. Write in decimal (base 10) notation the numbers 10101_2, 10101_3, 211_4, 126_8.
8. Write the number 100_{10} in base 2, base 3, base 4, base 5, base 6, base 7, base 8, base 9.
9. Count to 100_3 in base 3.
10. Write down the addition and multiplication tables in base 3.

3 Base 5

1. Convert these numbers to base 5:
   (a) 16
   (b) 124
   (c) 3
2. Convert these numbers from base 5 to base 10:
   (a) 121_5
   (b) 33_5
   (c) 2_5
3. Count to 100_5 in base 5.
4. Here is a multiplication table in base 10.

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

5. Now fill out a multiplication table in base 5. So you notice any patterns?

	1	2	3	4	10
1					
2					
3					
4					
10					

4 Base 12

1. How can we write numbers in a base greater than 10, like 12? How many symbols do we need for the digits?
2. Convert these numbers to base 12. You can use a $1 \leftarrow 12$ machine.
   (a) 12
   (b) 42
   (c) 58
   (d) 135
   (e) 144
   (f) 2012
3. Count to 100_{12} in base 12.
4. Convert to base 10:
   (a) 45_{12}
   (b) AB_{12}
   (c) $10A_{12}$

Extra Problems

5. Calculate
   (a) $248_{12} + 9A7_{12}$
   (b) $42_{12} \times 55_{12}$
6. Write down the multiplication table in base 12.

5 Alien Arithmetic

(a) Add $11121_3 + 122110_3$ (in base 3).

(b) Multiply 102_3 by 201_3 (in base 3).

(c) Calculate:

i. $341_5 + 203_5$

ii. $144_5 + 213_5$

iii. $413_5 - 22_5$

iv. $22_5 \times 31_5$

(d) Calculate

i. $1100_2 + 1101_2$

ii. $1011_2 - 101_2$

iii. $100011_2 - 10100_2$

iv. $1011_2 \times 101_2$

v. $10101_2 \div 11_2$

(e) Calculate

i. $101102_3 + 22012_3$

ii. $10120_3 - 212_3$

iii. $2012_3 \times 112_3$

Extra Problems

(f) Write down the addition and multiplication tables in base 4.

(g) Write down the addition and multiplication tables in base 5.

(h) In base 10, you can tell if a number is even based on whether or not its last digit is even. State and prove a condition (involving the representation of a number) that allows you to determine whether a number is odd or even

i. in the base 3 number system

ii. in the base n number system

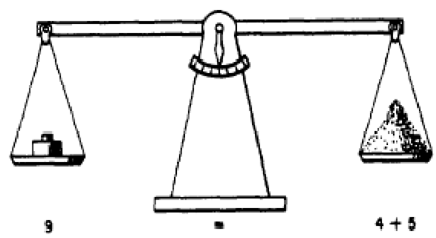
(i) Find and prove a divisibility rule in base 7 arithmetic that is analogous to the rule (in ordinary base 10 arithmetic) for divisibility by 9. See if you can find other divisibility rules in base 7 arithmetic that are similar to rules for base 10.

6 Aliens and Chocolate

1. You are a xeno-archeologist who has found an elementary school textbook from an ancient alien civilization. Although most of the book is no longer legible, you have found one equation that says: $3 \times 4 = 10$. How many fingers do you think the aliens have on each hand?

2. A spaceship full of hostile aliens is about to land on Earth. The aliens are very fond of Earth donuts, and you have persuaded them to leave you in peace in exchange for one donut for each alien on board. The captain radios down and says: "There are 100 of us total on board, and we would like 24 jelly donuts and 32 chocolate donuts with sprinkles." What number system is the alien using and how many donuts do you need to bring them?

3. Using a balance scale, you must be able to balance every whole kilogram amount of chocolate from 1 kg through 15 kg. You may choose four standard weights to use, each a different number of kilograms. Which weights should you use? (Weights may be placed only on the left pan, and chocolate only on the right.)



4. What is the smallest number of weights you need, so that you can weigh any whole number of grams of chocolate from 1 to 100 on a balance scale? Weights may be placed only on the left pan, and chocolate only on the right.

5. What if you are allowed to place the weights on either pan?

Extra Problems

6. Does there exist a number system where the following equations are true simultaneously?
   (a) $3 + 4 = 10$ and $3 \times 4 = 15$?
   (b) $2 + 3 = 5$ and $2 \times 3 = 11$?

7. A blackboard bears a half-erased calculation exercise:

$$\begin{array}{cccccc} & 2 & 3 & ? & 5 & ? \\ + & 1 & ? & 6 & 4 & 2 \\ \hline & 4 & 2 & 4 & 2 & 3 \end{array}$$

What number system was used and what are the missing digits?

Many of these problems are from *Mathematics Circles: the Russian Experience* by Fromkin, Genkin, and Itenberg

7 Magic Tricks

1. Birthday trick: We will present 5 cards with numbers on them. A volunteer will point to the cards that contain the date of their birthday. We will figure out their birthday.

2. 27 card trick. The magician uses 27 cards, and has a volunteer pick out their favorite card and also a favorite number between 1 and 27. The magician lays down cards in three piles, asks the volunteer to point to the pile that contains their card, and picks up the three piles. The magician then repeats this process: they lay down cards in three piles again, have the volunteer pick the pile that now contains the card, and picks up the three piles. The magician repeats this process a third and final time. Now, the magician counts out the favorite number of cards, and with luck and magic, the favorite card will be laid out as the last card counted out. For details, google ”Numberphile - 27 card trick” or follow this link: https://www.youtube.com/watch?v=l7lP9y7Bb5g