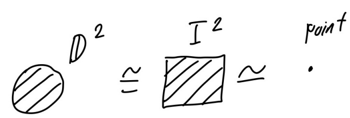
Topology of Surfaces and the Euler Characteristic

April 12, 2025

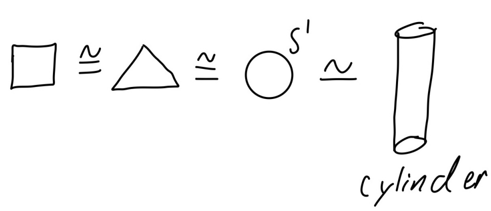
1 Introduction

The area of math known as *topology* studies shapes and properties they retain under continuous deformations. What is meant by continuous deformation can vary depending on context but typically we allow stretching, shrinking, bending, sometimes squishing, but never tearing or gluing together parts of the shape. If one shape can be turned into another using stretching, shrinking, and bending, we say they are *homeomorphic*, denoted $\cong$, and usually think of the two shapes as being the same, topologically. If one shape can be turned into another using stretching, shrinking, bending, and squishing, we say they are *homotopy equivalent*, which will be denoted $\simeq$. Because topology is like geometry but allowing for non-rigid transformations like bending, it is sometimes called 'rubberband geometry'.

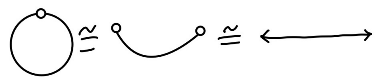
Here are some examples of homeomorphic and homotopy equivalent spaces: The closed unit disc $\mathbb{D}^2$ is homeomorphic to the filled-in unit square I^2 and both are homotopy equivalent to a single point:



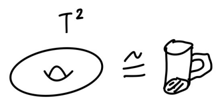
The circle S^1 is homeomorphic to any polygon and homotopy equivalent to a cylinder:



The circle S^1 minus any point is homeomorphic to the real number line $\mathbb{R} = (-\infty, \infty)$:

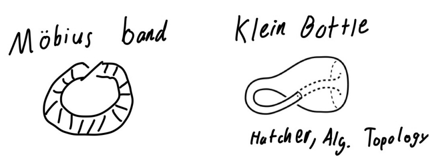


A donut is homeomorphic to a coffee mug:



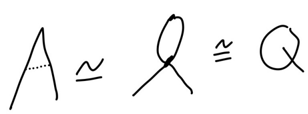
Topologists usually refer to the donut space as the *torus* T^2.

Two more exotic examples of topological spaces are the Möbius strip and the Klein bottle K :

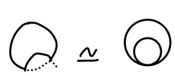


As a warning, the Klein bottle might look like it is self-intersecting, but that is only because I am being forced to draw a 2d picture of it. If we were able to draw in 4d, it would be possible to draw a non-self-intersecting Klein bottle.

Viewing the letter A as a shape, it is homotopy equivalent to the symbol Q by squishing the dotted line as seen below:



The previous example is dependent on how we represented the letters A and Q. Writing ‘Q’ as Q we would not have $A \simeq Q$ but instead Q would be homotopy equivalent to the *wedge of circles* $S^1 \vee S^1$ by squishing along the dotted line below:



Exercise 1. *For the letters A, B, C, D, K, L, O, Q, R, S, and T, which do you think are homeomorphic? Which do you think are homotopy equivalent?*

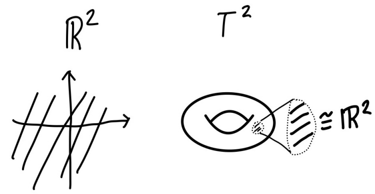
Frequently, it is easy to see that two spaces are homeomorphic or homotopy equivalent, since all you have to do is show how to start from one and do the appropriate squishing or stretching to get the other, but showing that two spaces are distinct is hard since you must show that now matter how you stretch or bend one you will never get the other. The solution to this problem is find topological *invariants* of spaces, which are properties or numbers associated to the space that are retained under homeomorphisms and/or homotopy equivalences. A good example of an invariant of a space is the number of *connected components.* The spaces 'i' and 'j' both have two connected components and '%' has three, whereas the circle S^1 has one. A space with one connected component is said to be *connected.* It is possible for spaces to have the same number of connected components while not being homeomorphic:

Exercise 2. *Show 'A' and 'Q' are not homeomorphic.*

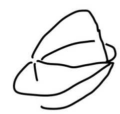
Exercise 3. *For the same letters as Exercise 1, which can you say are not homeomorphic?*

2 Surfaces and Getting New Spaces from the Old

An important class of topological spaces are *surfaces.* A surface M (without boundary) is a 2-dimensional spaces that locally looks flat. More precisely, at every point on the surface, there is a small region containing that point that is homeomorphic to the xy-plane $\mathbb{R}^2$. For example, the xy-plane $\mathbb{R}^2$ and the torus T^2 :



Some non-examples of surfaces include the circle S^1 and the following space that looks like two discs that have been glued along their diameters:



A small generalization to the definition of a surface is a surface with boundary. A surface with boundary is almost the same as a surface without boundary, the difference is that for a surface with boundary you are allowed to have points that are homeomorphic to the upper half-plane $\mathbb{H}$. For example, the cup C, which is homeomorphic to a space from section 1, is a surface with boundary because of the points at the rim:

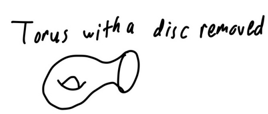
The points of M which locally look like the upper half-plane are called the *boundary* of M and denoted ∂M.

Exercise 4. *Which spaces from section 1 are surfaces (with or without boundary)? Can you name any other examples of surfaces?*

Exercise 5. *Is the property of being a surface an invariant up to homotopy equivalence? That is, if two spaces are homotopy equivalent and one is a surface, must the other be a surface?*

Exercise 6. *Is being a surface invariant up to homeomorphism? That is, if two spaces are homeomorphic and one is a surface, must the other be a surface?*

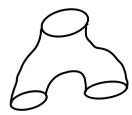
Given any surface, we are able to create a new surface withh boundary by choosing some point, finding an open disc around that point, and removing it from the surface.



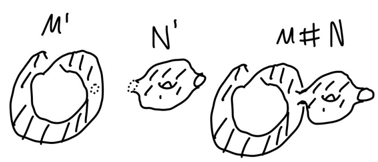
Exercise 7. *What space do you get when you remove an open disc from the sphere S^2?*

On the other hand, if you have a surface with boundary and the boundary contains a copy of the circle S^1, you may glue in a copy of the disc $\mathbb{D}^2$ along a copy of S^1 to get a surface, potentially with boundary.

Exercise 8. *Glue a disc to the pair of pants space below and determine what space you got a homeomorphic copy of. What if you glue two discs? Three?*



Given two surfaces M and N, we may form their *connect sum*, written $M\#N$, by choosing some points p on M and q on N that are not on their respective boundary, removing an open disc around p and q and gluing the two resultant surfaces with boundary along the newly created boundary circles. Below we see this process for $M =$ the Möbius strip and $N = T^2$ with a disc removed. M' and N' are M and N after having removed copies of the disc and we glue them along the dotted lines.



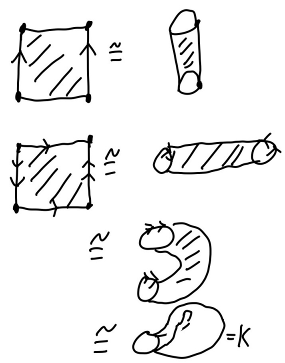
Exercise 9. *Let M be any surface. What is $M\#S^2$ homeomorphic to?*

Exercise 10. *Draw $T^2\#T^2$ and $T^2\#T^2\#T^2$. Can you guess what $\underbrace{T^2\#T^2\ldots\#T^2}_{g\ times}$ looks like?*

The space $M_g := \underbrace{T^2 \# T^2 \dots \# T^2}_{g \text{ times}}$ is called the *genus g torus*.

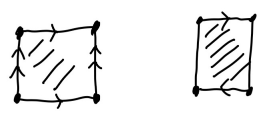
3 Gluing Polygons and the Euler Characteristic

One way to construct surfaces is by gluing together the edges of a polygon according to some given pairing of edges. For example, the two below give a cylinder and the Klein bottle:

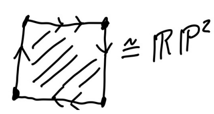


Note that when we had unpaired edges we got a surface with boundary.

Exercise 11. *What surfaces do the following gluing diagrams represent?*



One surface that we have not been able to describe until now is $\mathbb{RP}^2$, the *real projective plane*. There is no good way to show what it looks like, but it may be described as the set of lines in xyz-space that go through the origin. It is given by the following gluing diagram:



The m-fold projective plane is $\Sigma_m := \underbrace{\mathbb{RP}^2 \# \mathbb{RP}^2 \dots \# \mathbb{RP}^2}_{m \text{ times}}$

Thus far, all of polygons have been squares, but there is nothing preventing us from having polygons with more or fewer sides:

Exercise 12. *Determine what surface is given from each of the above gluing diagrams.*

For any topological space X that can be built up by gluing together vertices (points), edges (line segments), and faces (discs), we may define the Euler characteristic $\chi(X) = V - E + F$, where V is the number of vertices, E the number of edges, and F the number of faces. Note that you count how many of each type of piece there is after the gluing is done. Thus the Klein bottle has $V = 1, E = 2, F = 1$, so $\chi(K) = 1 - 2 + 1 = 0$.

Exercise 13. *We have seen two gluing diagrams that give rise to the Möbius strip. Using both of them, determine the Euler characteristic of the Möbius strip. What do you notice?*

In fact, the Euler characteristic of a space is invariant up to homotopy equivalence.

Exercise 14. *Determine the Euler characteristic of a point, the circle* S^1*, the sphere* S^2*, the disc* $\mathbb{B}^2$*, the torus* T^2*, and the real projective plane* $\mathbb{RP}^2$*.*

Exercise 15. *If* N *is the surface* M *after removing a disc, what is the relationship between* $\chi(N)$ *and* $\chi(M)$*?*

Exercise 16. *Given two surfaces* M *and* N*, what is the relationship between* $\chi(M), \chi(N)$*, and* $\chi(M\#N)$*?*

Exercise 17. *Find* $\chi(M_g)$ *and* $\chi(\Sigma_m)$ *for* $g > 0$ *and* $m > 0$*. What do* $T^2 = M_1$ *and* Σ_2 *have in common?*

4 Orientability and the Classification of Surfaces

We have seen that the Euler characteristic cannot distinguish between the torus and the 2-fold real projective plane. This is about as bad as the situation can get however. We say a surface M is non-orientable if it contains a copy of the Möbius strip and is orientable if it does not.

Exercise 18. *Is T^2 orientable? Is $\mathbb{RP}^2$ orientable?*

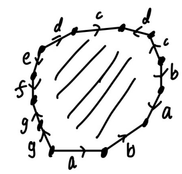
Exercise 19. *Suppose M is non-orientiable. Can anything be said about the orientability of $M\#N$ for another surface N? What if both M and N are orientable instead?*

We say a surface is closed if it has no boundary and satisfies a smallness condition known as compactness. Defining compactness is hard, but we wish to state the following theorem properly. Any surfaces considered from this point on will be compact.

Theorem 1 (Classification of Surfaces)**.** *Suppose M is a closed surface. Then*

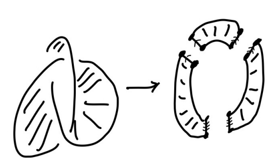
- $\chi(M) \leq 2$ *with* $\chi(M) = 2$ *if and only if* $M \cong S^2$,
- *If* $\chi(M) = 2 - g$ *for some odd number* g, *then* M *is non-orientable and* $M \cong \Sigma_g$,
- *If* $\chi(M) = 2 - g$ *for some even number* $g > 0$ *and* M *is non-orientable then* $M \cong \Sigma_g$, *and*
- *If* $\chi(M) = 2 - 2g$ *and* M *is orientable then* $M \cong M_g$.

This lets us easily determine the surface given by the gluing diagram, which has Euler characteristic 1 - 6 + 1 = -4 and is orientable must give the genus 3 torus M_3:



Exercise 20. *How can you use the classification of surfaces to identify a surface with boundary?*

For example, the following knot diagram determines a surface with boundary. We can see that it has one boundary component because the knot itself is connected. It is non-orientable because there is a loop that goes through an odd number of half-twists. We can see that the Euler characteristic of the surface with a disc removed is 0.



Exercise 21. *What are the surfaces determined by the following two link diagrams?*