Chapel Hill Math Circle
Session 11 – March 1, 2025:

Spaghetti and Pizza[1] Numbers Part II

Beginners' Group (grades 1-3), 10:30-11:30a
Mr. Barman – dilip@trianglemathinstitute.com

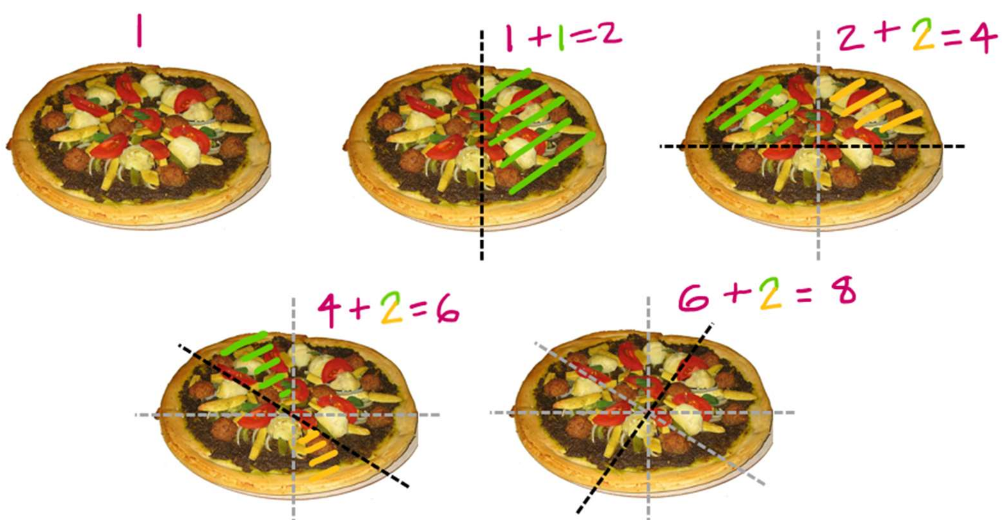
Supplies needed per student: 1-2 white paper plates, at least 6-8" in diameter (alternatively several circles drawn on paper), straightedge (like a ruler), as well as paper and pencil

Welcome to Math Circle session 11. If you were here last time we learned about pieces that you can make with straight cuts of either spaghetti or pizza. We will review this then move on – it should make sense whether you were here last time or not.

Cutting spaghetti, pizza, and uttapam

Last time we experimented with cutting spaghetti and circles representing pizza or uttapam (an Indian crepe – which we actually got from Vimala's Curryblossom Café, cut and ate! You can see the picture of one of our cutting attempts of a rectangular uttapam above and even see a video[2], both thanks to Ms. Victoria[3].). The goal was to use straight line cuts and get the most pieces possible, even if the pieces were rather odd looking.

We easily figured out the "Spaghetti Numbers"; how many pieces of spaghetti you would get without moving pieces around. We start with 1 piece with 0 cuts; every cut creates a new piece.



We thought about the typical way to cut pizza as wedges, as shown here. Then we thought about how to cut pizza if the goal is to make the most pieces possible. We found some surprises!

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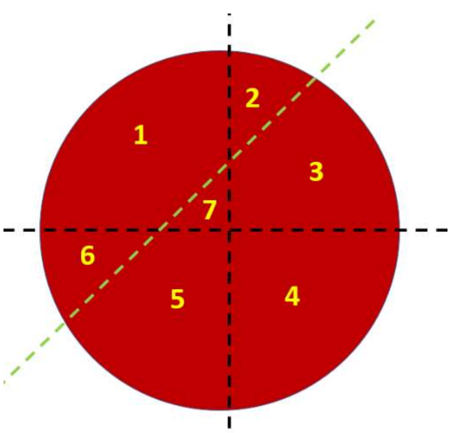
[1] The pizza pictures are of a vegan pesto pizza that I made on April 25, 2006.
[2] bsky.app/profile/dbarman.bsky.social/post/3liabntb4fc2i
[3] Thanks to Vimala Rajendran of Vimala's Curryblossom Café for her special order, Nav Dessai for picking up the uttapam, and to Victoria Yan for the picture

Ava, a Stanford Math Circle 1st grade student of mine, made this table in late February 2025. Take a minute and be sure that you agree with the Spaghetti Numbers and Pizza Wedge Numbers.

The last column is just doubling numbers; if we line pieces up after cutting them, we can always double the number of pieces that we have, right?

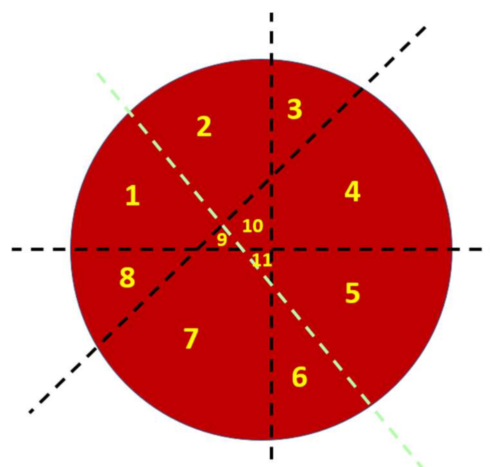
Now look at her "MPW" column. We found last time that with 0 cuts we have 1 pizza, with 1 cut we have 2 pieces, and with 2 cuts we have 4 pieces. But we can actually have 7 pieces with 3 cuts if we cut this way:

Straight cuts without rearranging				... and with rearranging
Number of Straight Cuts i	# of Spaghetti Pieces S_i	# of Pizza Wedge Pieces PW_i	Maximum # of Pizza Pieces MPW_i	# of Spaghetti or Pizza Pieces $SPWR_i$
0	1	1	1	1
1	1+1=2	1+1=2	Δ1+1=2	2
2	2+1=3	2+2=4	Δ2+1=4	4
3	3+1=4	3+3=6	Δ3+1=7	8
4	4+1=5	4+4=8	Δ4+1=11	16
5	5+1=6	5+5=10	Δ5+1=16	32
6	6+1=7	6+6=12	Δ6+1=22	64
7	7+1=8	7+7=14	Δ7+1=29	128
100	100+1=101	100+100=200	~~Δ8+1=37~~ Δ100+1=5051	2^{100}
n	n+1	n+n	Δn+1	2^n



Then we can get 11 pieces by cutting like this:

Take a few minutes and look at the sequence of Pizza Numbers so far: 1, 2, 4, 7, 11. What is the pattern? Can you predict the next few numbers in the sequence?



This is how Ava explained the patterns we see. What do Triangle Numbers have to do with cutting pizza?

If you need a review, Triangle Numbers look like 0, 1, 1+2=3, 1+2+3=6, 1+2+3+4=10, 1+2+3+4+5=15, and so forth. On the bottom of this page is a diagram of the first few Triangle Numbers skipping 0; we say $T_1=1$, $T_2=1+2+3$, and so forth. Do you remember how to find the 10th Triangle Number? And what do these numbers have to do with pizza cutting?

Explanations:

1. Spaghetti – After every cut, you get 1 extra piece So after 1 cut you get 1+1=2 pieces and then after ~~you~~ cut 2, you get 2+1=3 pieces and so forth and that is why you get n+1 pieces after n cuts.

2. Pizza Wedges – For Pizza Wedges when we cut through the center for every cut we get 2 extra pieces. For cut 1, we get 1+1=2 pieces. For cut 2 we get 2+2=4 pieces. This includes the 2 extra pieces from this cut. For 3 ~~pieces~~ cuts we get 3+3=6 pieces. Now we see a pattern, where the number of pieces is double the number of ~~pieces~~ cuts. That is why we get n+n pieces after n cuts.

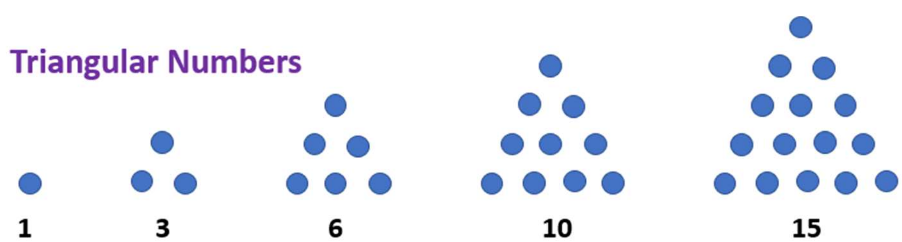
To get maximum wedges we need to make straight cuts through all the previous cuts. Here's how it works: for cut 0 we get one piece – the pizza itself! for cut 1 we get two pieces. For cut 2 we need to cut through these 2 pieces to get 4 pieces. For cut 3 we have to cut through 3 regions to get 7 pieces. This follows a pattern:

$C_0=0$ $P_0=1$
$C_1=1$ $P_1=2$ or 1+(1) →
$C_2=2$ $P_2=4$ or 1+(3) → These are Triangle numbers!
$C_3=3$ $P_3=7$ or 1+(6) →
$C_4=4$ $P_4=11$ or 1+(10) →

This means for n cuts it is the triangle n+1 or $P_n = \Delta_{n+1}$

With rearranging, its easy. Every time we cut we can rearrange the pieces so that the cut goes through all of the pieces and they double. This is true for: Spaghetti, Pizza and even watermelons!

Triangular Numbers



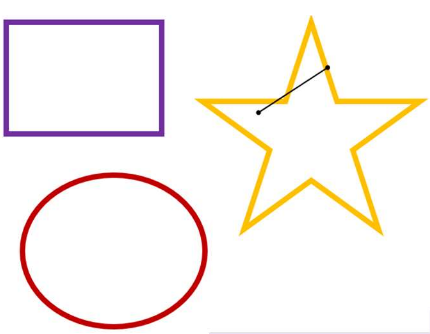
$P_n = \Delta_n + 1$?

Here's the first 5 Pizza Numbers: 1, 2, 4, 7, 11

Here's the first 5 Triangular Numbers: 0, 1, 3, 6, 10

It looks like my proposition that a Pizza Number is one more than the corresponding Triangle Number is true. Is it? Use logic to argue this. Take a few minutes to draw some circles representing pizza and straight lines, and talk with a partner. Does it make sense that $P_1 = \Delta_1 + 1$, $P_2 = \Delta_2 + 1$, and in fact that $P_n = \Delta_n + 1$ for any counting number that you pick? Think about what happens when you cut through an old slice and when you reach the crust at the far end.

Let's discuss what you came up with. Don't peek but I have a page that summarizes Spaghetti and Pizza Numbers and explains the patterns on the next page. To the right you can see a screen capture from one of the videos that Ms. Victoria had taken from last time; this is when I was cutting into the typical round uttapam.

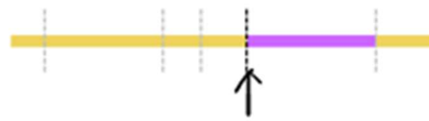


Would you get the same numbers if you cut any convex shape? The yellow star here is concave; you can find a straight line connecting two points inside that has to leave the shape. The other two shapes are convex.

Here is my table with explanations

Straight Cuts without Rearranging (with Rearranging the pieces just double)			
Number of Straight Cuts i	Pieces of Spaghetti S_i	Pieces of Pizza Wedges M_i	max Pieces of Pizza P_i
0	1	1	$T_0+1 = 1$
1	1+1 = 2	1+1=2	$T_1+1 = 2$
2	2+1 = 3	2+2 = 4	$T_2+1 = 4$
3	3+1 = 4	3+3 = 6	$T_3+1 = 7$
4	4+1 = 5	4+4 = 8	$T_4+1 = 11$
5	5+1 = 6	5+5 = 10	$T_5+1 = 16$
6	6+1 = 7	6+6 = 12	$T_6+1 = 22$
100	100+1 = 101	100+100 = 200	$T_{100}+1 = 5051$
n	n+1	n+n	T_n+1

The pieces of spaghetti always increase by one because with every cut, one new piece is created as shown, so in n cuts we get **n+1** pieces.



When we cut wedges, we always cut through one piece on either side of the center; each of those two slices now results in a new slice. Therefore with n cuts we end up with **n+n** slices.

When we are trying to get the most pieces, we want to cut through all of the existing lines. On cut n, we cut through n-1 lines but also the crust, so we get n-1 + 1 = n new pieces. With 0 cuts we have 1 piece. With 1 cut we have 2; let's write this as 1 plus a new 1 or 2. With 2 cuts we have the old 2 plus 1+1 (going through 1 old cut and the crust) = 4. These all match the pattern T_n+1. Assuming that is true for the first n-1 cuts, on cut n we get $(T_{n-1}+1) + 1+(n-1) = (T_{n-1}+1)+n = T_n+1$. For n cuts we just **add 1 to Triangle Number n.**

The cases with rearranging are clearly just doubling as we can move our pieces so that the cutter splits each piece, whether spaghetti or pizza (or watermelon).

Theory vs. Practice

Last time we discussed why you may not get the number of pieces that you predict. Do you remember why theory and practice may be different? In theory I can get 22 pieces with 6 straight cuts but I very much doubt that I actually would in practice. What do you think? You can try by using a circle, a paper plate, or actual pizza (or uttapam) and cutting or drawing straight lines.

Cuts	0	1	2	3	4	5	6
Pieces	1	2	4	7	11	16	22

Let's look at a cool Numberphile video *The Lazy Way to Cut Pizza*[4] by math teacher Tom Crawford[5]. Dr. Crawford tries to make the 22 pieces that we should be able to get with 6 straight cuts (T_6+1 = 1+2+3+4+5+6 + 1 = 21+1 = 22) but only gets 18. Let's watch a segment from about 2:26-4:12 where he discusses the theory and then from about 10:00-12:20 when he tries cutting an actual pizza.

Watermelon Numbers

We won't have time in class but at home I encourage you to try cutting a solid three-dimensional object. In the summer watermelon might be a good choice; the picture here shows Dr. Green visiting a camp that I teach at the NC School of Science and Math[6]. Notice how after each cut Professor Green wraps up the watermelon with tape before making the next cut.

Instead of watermelon something that my students have found works well is Play-Doh® or maybe clay. You could try a grapefruit or something else. With an adult's help try making straight cuts, reassembling, and cutting again. The first few Watermelon Numbers are 1, 2, 4, 8, 15, 26, 42, and 64.

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[4] youtube.com/watch?v=Xd9UZSodeN8

[5] tomrocksmaths.com

[6] I took the picture on July 21, 2021. Professor Linda Green from the University of North Carolina led us on a session where inside we figured out Spaghetti (she used licorice), Pizza, and Watermelon Numbers. Outside we tried cutting watermelon and checked to see if we got the number of pieces that we expected.

Spaghetti, Pizza, and Watermelon Numbers without rearranging

I have summarized below these numbers. Take a few minutes and see if you can find patterns. How do Spaghetti, Pizza, and Watermelon Numbers depend on earlier values?

Number of Straight Cuts i	Spaghetti (G_i)	Pizza (P_i)	Watermelon (W_i)
0	1	1	1
1	2	2	2
2	3	4	4
3	4	7	8
4	5	11	15
5	6	16	26
6	7	22	42
10	11	56	
20	21	211	
30	31	466	
40	41	821	
n	n+1	$T_n + 1$	

We could and may do even more but this will probably bring us to the end of our hour. Here are a few activities that you can try till we meet again. Try (carefully and with an adult) cutting a solid like watermelon or clay. Can you find a pattern that will let you figure out Watermelon Numbers? Do you remember Pascal's Triangle? I hope soon to talk about Bernoulli's Triangle; it's just a running total of each row (e.g., the row 1 3 3 1 becomes 1 1+3 1+3+3 1+3+3+1 or 1 4 7 8). What patterns do you see?

Have Fun!
Mr. Barman

Notes for Parents

Recurrence relations

I was quite impressed in my Stanford Math Circle that many students found the recurrence relation $Pn = Pn-1 + n$ and $W_n = W_{n-1} + P_{n-1}$ and bet that either children today will find these or if they had a bit more time they would. If you haven't seen this term before, it simply describes a sequence in terms of previous items in the sequence. A trivial (and not very useful) example is the i^{th} natural number; it could be defined as the $(i-1)^{st}$ one plus 1. More completely, including a "stopping rule" to unwind our recurrence and using a triple bar to mean "is defined as", we have:

$$\aleph_1 \equiv 1$$

$$\aleph_i = \aleph_{i-1} + 1 \quad \forall i \epsilon \mathbb{Z} > 1$$

Of course this is overkill (a much simpler recurrence would simply be that the i^{th} natural number is i for all positive integers i), but an opportunity to describe some terminology. What this says is that the 1^{st} natural number is defined as 1. Now pick any integer greater than 1 and call it i. This i^{th} natural number is just the previous one plus 1.

I almost feel like Alice from *Alice in Wonderland* in that we are discussing such building blocks of more advanced math in the 1^{st} and 2^{nd} grade. The fact that children came up with recurrences excites me for many reasons. Even for those who didn't get such a recurrence, I am confident that in today's class they will have seen many pattern. The fact that 1^{st} and 2^{nd} graders can extrapolate results to indexed variables is awesome.

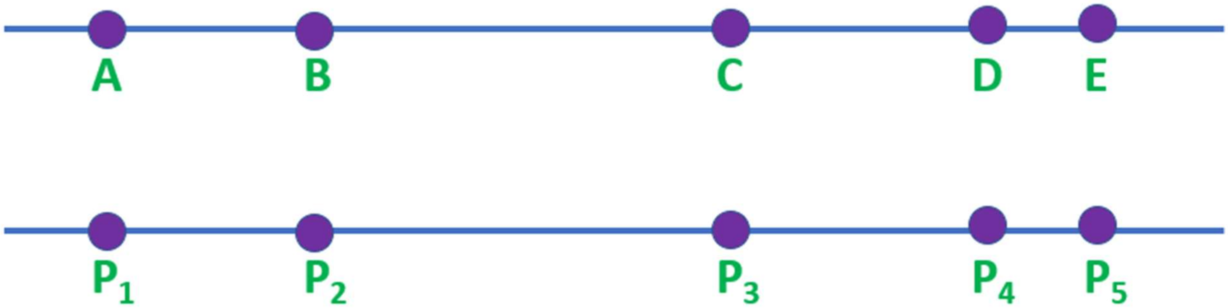
Here's some of what thrills me about this:

- Wow, I don't think I really got formally introduced to representing and manipulating recurrence relations till upper-level college classes. The concept is straightforward and our playing here has, if nothing else, planted a seed for terminology that students can embrace or start to open up their cognition eventually to readily use. I often find students perhaps in prealgebra or beyond who stumble on terminology and formalities in math – if only they were exposed to these kinds of ideas in a fun and enticing manner, I believe that many or most of them would imbibe these concepts much more readily.
- Even more basic is the ease with which the students seem to be accepting and using subscripts. In some educational settings the idea of a variable isn't really introduced until 6^{th} grade or beyond. I see no reason not to start talking about variables in 2^{nd} grade – and these students show that even 1^{st} graders can understand the whys and hows of such an abstraction.
- The idea of a recurrence can help a student see implicit structure in data and ready them for other ideas like recursion, important in areas such as programming; algorithm design; sorting out logic and illogic in areas like self-reference and even solipsism (more on that when we discuss *Flatland* below); and even epistemology, self-knowledge, and philosophy.

Subscripted variables

If you want to solidify the idea of a subscript, try this. Ask them to draw a line representing a straight road. Now ask them to put drawings of houses along the "road" and put the names of friends on the houses.

Efficiency is important in math – it helps us home in on problems and solve them more quickly. So we can "improve", at least in some sense, the diagram by not using names like "Fred", "Sapna", "Tyrone", etc., but maybe with letters – variables. And we can dispense with the houses and simply draw points. Perhaps your child now has a diagram like the first one below. It's an easy argument from this to the second diagram to point out that we have trouble if we have more than 26 names; sure, we can use other symbols, but subscripts make a lot of sense. We can now index through a set of data. Powerful – for 1st and 2nd graders!!



I'd love to see pictures or videos of any cutting experiments that you do. Thanks for coming to Chapel Hill Math Circle.

Till next time!

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