# Chapel Hill Math Circle Exit Ticket:
## March 1, 2025: Inversive Geometry

Please remove this sheet, complete it, and return it before the end of our session.

- Did you find today's topic interesting?
- Was the this topic appropriately challenging relative to your background? That is, was the topic neither too elementary nor inaccessibly advanced?
- How could we improve this worksheet for future sessions?
- What did you enjoy about today's topic?
- What did you find particularly challenging?
- Was there anything you thought was too difficult?
- Was there anything you thought was too easy?
- Are there any topics you would be interested in seeing us cover in the future?

# Inversive Geometry

**Abstract**

In this session, we shall explore *inversive geometry*, also called *circle inversion*, *circular inversion*, *reflection about a circle*, *inversion with respect to a circle*, or, when the context is clear, simply *inversion*. Beginning with a fixed circle $\odot O$ in the plane having center $O$, inversion with respect to $\odot O$ essentially turns the plane inside out. We consider some examples of inversion, explore the inversion's properties, and use these methods to prove theorems and solve problems taken from mathematics competitions.

*Background needed:* Prerequisites include familiarity with geometry and algebra, especially properties of circles in the plane. Familiarity with similar triangles, in particular, will be helpful for some exercises.

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# Contents

# 0 Warmup

As prerequisites for this session, it will help to answer the following first. You do *not* need to know these answers already, and many questions will be revisited later in the worksheet.

0.1 In the plane, let $\odot O$ be a circle with center $O$. What does it mean for a line $\ell$ to be *tangent* to $\odot O$? If $\odot O'$ is another circle, what does it mean that $\odot O'$ is tangent to $\odot O$?

0.2 What are *similar polygons*? In particular, if $\triangle ABC$ is similar to $\triangle DEF$ (with vertices in that order!), denoted $\triangle ABC \sim \triangle DEF$, what does this mean? Conversely, if we want to prove that $\triangle ABC \sim \triangle DEF$, how might we do so?

# 1 An Introduction to Inversion

## 1.1 Discussion

To motivate our definition for inversive geometry, let's first consider an algebraic version of inversion to motivate what we shall do geometrically. Consider a function $f$ defined by the formula

$$f(x) := \frac{1}{x}, \tag{1.1.1}$$

so that $f$ is the function that sends the number $x$ to its reciprocal $1/x$. Then $f$ has the following familiar properties:

**Proposition 1.1.1.** *If $f$ is a function defined by (1.1.1), then we have the following:*

*(a) For all $x$, $f(x)$ is defined if and only if $x \neq 0$.*

*(b) For all $x$ such that $f(x)$ is defined, $f(f(x)) = x$.*

*That is, the reciprocal* of *the reciprocal of a number $x$ is the original number $x$ itself.*

*(c) If $x$ is a positive real number, then $f(x) = x$ if and only if $x = 1$.*

*Equivalently, $x = 1$ is the unique positive real number such that $1/x = x$.*

*(d) If $x$ is a positive real number, then $x < 1$ implies $f(x) > 1$, and $x > 1$ implies $f(x) < 1$.*

*That is, $f$ sends small numbers to large ones, and vice versa.*

(e) *The properties above can be extended from the positive real numbers to the "punctured plane" of complex numbers*[1] *excluding* $0$*, a set denoted* $\mathbb{C}\setminus\{0\}$*. If* $z$ *is a complex number, then* $f(z) = 1/z$ *is defined if and only if* $z \neq 0$*. If* $|z|$ *denotes the length of* $z$*, then* $|z| = 1$ *if and only if* $|f(z)| = 1$*,* $|z| < 1$ *if and only if* $|f(z)| > 1$*, and* $|z| > 1$ *if and only if* $|z| < 1$*.*

Our goal is to describe a geometric transformation that shares many of the properties in Proposition 1.1.1.

**Definition 1.1.2.** Let $O$ be a point in the plane, and let $\odot O$ denote the circle with center $O$ and radius $r > 0$. If $P$ is a point in the plane not equal to the center $O$, then the *inverse of* $P$ *with respect to* $\odot O$ is the unique point $P'$ on the ray $\overrightarrow{OP}$ such that

$$OP \cdot OP' = r^2. \tag{1.1.2}$$

The inverse of the point $O$ itself with respect to the circle $\odot O$ is not defined.[2]

*Notation.* Let $\odot O$ be the circle as in Definition 1.1.2, which we call our *reference circle*. If $P \neq O$ is some point in the plane, we shall typically denote the inverse of $P$ with respect to $\odot O$ by $P'$. When images are color coded, our initial points $P$ will typically be colored blue, and the inverse points $P'$ will be colored red, while the reference circle itself will typically be colored black.

See Figure 1.1.1 for examples. Our goal in Definition 1.1.2 will be to establish properties similar to those in Proposition 1.1.1. Later, we shall explore additional properties unique to inversion that go beyond Proposition 1.1.1.

## 1.2 Exercises

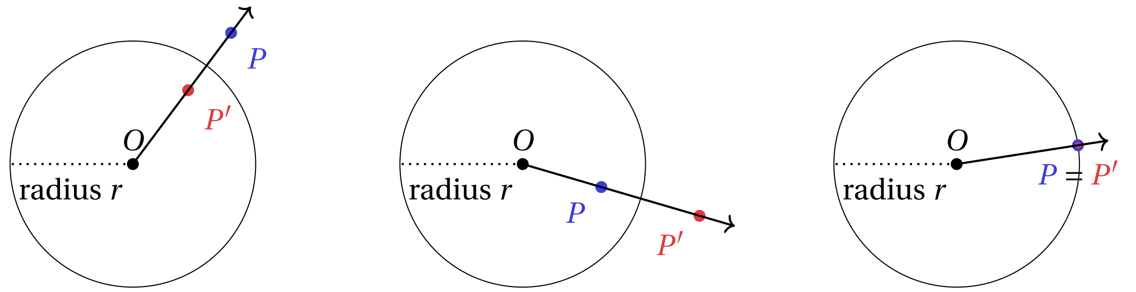
Throughout, $O$ will denote a point in the plane, $\odot O$ will denote a fixed reference circle for inversion, and we will have initial points $P \neq O$ and their inverses $P'$, as in our notation convention after Definition 1.1.2.

1.2.1 Consider the reference circle $\odot O$ and given points $P_i$ in Figure 1.2.1. Using Definition 1.1.2, determine where the inverse points $P_i'$ should be placed in the figure.

*Note:* The goal here is to find the *general* locations of the points $P_i'$ in terms of the given points $P_i$ and the reference circle. Loose approximations and estimates are fine, rather than computations with spurious precision.

[1] That is, $z$ is a number of the form $z = a + bi$, where $a, b$ are real numbers, and $i$ is the imaginary unit such that $i^2 = -1$.

[2] In other settings, one defines the inverse of $O$ itself to be the "*point at infinity*", a perfectly rigorous notion for a point in the *projective plane*. For our purposes, though, we'll take the convention that the inverse $O'$ of $O$ is not defined, while noting that the behavior of the inverses of points *near* $O$ is that these inverses get further and further from $O$.




(a) Defining $P'$ when $P$ is outside $\odot O$: $OP \cdot OP' = r^2$.

(b) Defining $P'$ when $P$ is inside $\odot O$: $OP \cdot OP' = r^2$.

(c) When $P$ is on $\odot O$, $P = P'$, and $OP = OP' = r$.

Figure 1.1.1: For a point $P \neq O$ in the plane we define $P'$, the inverse of $P$ with respect to $\odot O$.

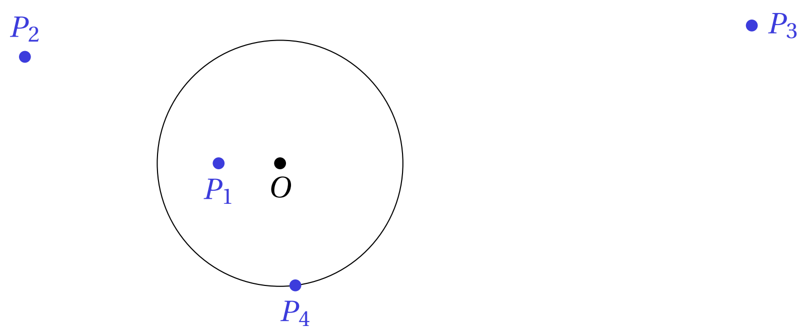



Figure 1.2.1: Computing the inverse points $P_i'$ for the given points $P_i$, relative to the reference circle $\odot O$.

1.2.2 Let $P \neq O$ be a point in the plane with $P'$ the inverse of $P$ with respect to $\odot O$. Further, let $P''$ denote the inverse of $P'$ with respect to $\odot O$. Explain why $P'' = P$.

That is, the explain why the inverse of the inverse of $P$ is $P$ once again. Compare to Proposition 1.1.0(b) above.

1.2.3 Let $P \neq O$. Prove that $P' = P$ if and only if $P$ lies on $\odot O$. Compare this result to Proposition 1.1.0(c).

1.2.4 Assume that $P$ does not lie on $\odot O$. Prove that if $P$ lies outside $\odot O$, then $P'$ lies inside $\odot O$. Similarly, prove that if $P$ lies inside $\odot O$, then $P'$ lies outside $\odot O$. Compare this to Propositions 1.1.0(d)–1.1.0(e).

# 2 Exploring Properties of Inversion

## 2.1 Discussion

In Section 1, we defined inversion with respect to a reference circle $\odot O$, then explored some of the basic properties of this transformation of the punctured plane. Considering the image $P'$ of *one* point $P$, though, is an incomplete way to understand inversion. For deeper insight, we shall explore the following questions:

- If $\mathcal{R}$ is a region of the plane, what is the set $\mathcal{R}'$ of its inverses? We will be most interested when $\mathcal{R}$ is either another circle $\odot Q$ or a line $\ell$.
- If $P, Q \neq O$, what is the distance $P'Q'$ of the images of $P$ and $Q$ in terms of the length $PQ$?
- How can we apply what we learn about properties of inversion to solve other problems in geometry?

As with Exercise #1.2.1, we are more interested in a general understanding of the behavior of inversion rather than in answers that are precisely to scale.

## 2.2 Exercises

As in Section 1.2, unless otherwise indicated, we shall assume throughout that $\odot O$ is a fixed circle, our reference circle for inversion.

2.2.1 Let $Q$ be a point outside $\odot O$, and assume that $\mathcal{R} := \odot Q$ is a circle lying completely outside $\odot O$. What is the inverse of $\odot Q$ with respect to $\odot O$? See Figure 2.2.1 for reference.

2.2.2 Let $Q$ be a point inside $\odot O$, and assume that $\mathcal{R} := \odot Q$ is a circle lying completely inside $\odot O$. Further, assume that $\odot Q$ does *not* enclose the center $O$ of the reference circle. What is the inverse of $\odot Q$ with respect to $\odot O$? See Figure 2.2.2 for reference.

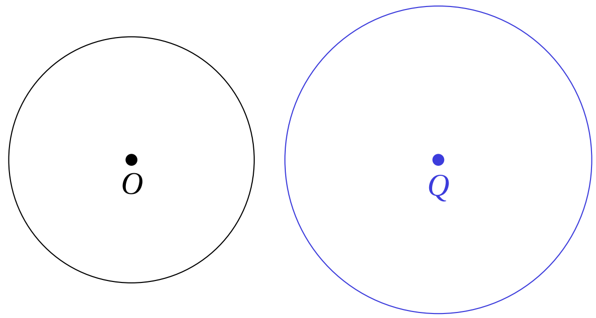



Figure 2.2.1: Computing the inverse of $\odot Q$ when $\odot Q$ lies entirely outside $\odot O$.

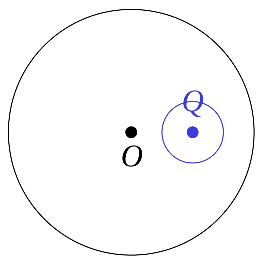



Figure 2.2.2: Computing the inverse of $\odot Q$ when $\odot Q$ lies entirely inside $\odot O$ and $\odot Q$ does *not* enclose $O$.

2.2.3 Let $Q$ be a point inside $\odot O$, and assume that $\mathcal{R} := \odot Q$ is a circle lying completely inside $\odot O$. Further, assume that $\odot Q$ does *not* enclose the center $O$ of the reference circle. What is the inverse of $\odot Q$ with respect to $\odot O$? See Figure 2.2.3 for reference.

2.2.4 Let $\odot Q$ be a circle that is tangent to $\odot O$ without enclosing $O$. What is the inverse of $\odot Q$ with respect to $\odot O$? See Figure 2.2.4 for reference.

*Note:* There are two possibilities for the position of $\odot Q$: either $\odot Q$ is externally tangent to $\odot O$, or $\odot Q$ is internally tangent to $\odot O$. Explain why it suffices to consider just one of these cases.

2.2.5 Let $\odot Q$ be a circle that intersects, but is not tangent to, $\odot O$, Further, assume $\odot Q$ does not enclose $O$. What is the inverse of $\odot Q$ with respect to $\odot O$? See Figure 2.2.5 for reference.

*Note:* There are two possibilities for the position of $\odot Q$: either $\odot Q$ is externally

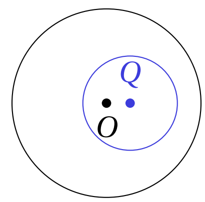



Figure 2.2.3: Computing the inverse of $\odot Q$ when $\odot Q$ lies entirely inside $\odot O$ and $\odot Q$ *does* enclose $O$.

tangent to $\odot O$, or $\odot Q$ is internally tangent to $\odot O$. Explain why it suffices to consider just one of these cases.

2.2.6 Let $\odot Q$ be a circle that passes through $O$, the center of the reference circle. What is the inverse of $\odot Q$? See Figure 2.2.6 for reference.

*Note:* There are several possibilities for the position of $\odot Q$. Either $\odot Q$ lies entirely inside $\odot O$, $\odot Q$ is internally tangent to $\odot O$, or $\odot Q$ passes through $O$ and two points of $\odot O$. Only one of these is shown in Figure 2.2.6, though, so you will need to consider these cases separately.

The inverse of $O$ itself cannot be defined, so you need not consider the inverse of this specific point.

2.2.7 Let $\odot O$ be a reference circle of radius $r$. If $P, Q \neq O$ are points in the plane with respective inverses $P', Q'$ with respect to $\odot O$, then compute the length $P'Q'$ in terms of $PQ$, $OP$, $OQ$, and $r$ by verifying that

$$P'Q' = PQ \cdot \frac{r^2}{OP \cdot OQ}. \tag{2.2.1}$$

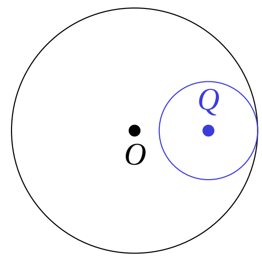



Figure 2.2.4: Computing the inverse of $\odot Q$ when $\odot Q$ lies entirely inside $\odot O$ and $\odot Q$ *does* enclose $O$. Shown: the case where $\odot Q$ is internally tangent to $\odot O$.

See Figure 2.2.7 for context.

*Hint:* By Definition 1.1.2 and Equation (1.1.2), we know how to compute $OP$ and $OQ$ in terms of $OP'$, $OQ'$, and $r$. Can you find any similar triangles to help you in your computations?

# 3 Using Inversion in Proofs and Problem-Solving

## 3.1 Discussion

Many of the exercises in Section 2.2 suggested the following conjecture:

**Conjecture 3.1.1.** *Call a region $\mathcal{R}$ in the plane a generalized circle if $\mathcal{R}$ is either a circle $\Omega$ or a line $\ell$. If $\odot O$ is a reference circle in the plane, then inversion with respect to $\odot O$ will map generalized circles to generalized circles, with the exception that "$O'$" is undefined, and there is no point whose inverse is $O$.*

In general, proving statements about circles can be quite challenging, whereas proving statements about line segments is often much simpler. More generally, we have the following general principle, taken from Figure 9 in [36]:

**Strategy 3.1.2.** *If given a problem in geometry, consider identifying a reference circle, inverting, since the inverted problem may be more tractable. This can be modeled in the following way:*

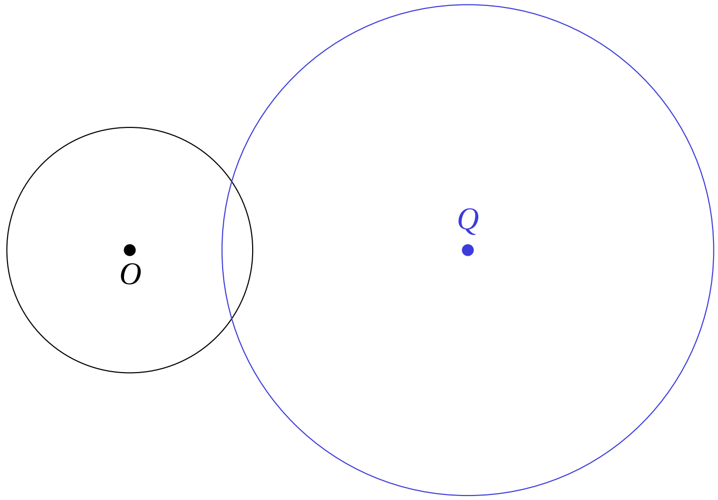



Figure 2.2.5: Computing the inverse of $\odot Q$ when $\odot Q$ intersects $\odot O$, $\odot Q$ is not tangent to $\odot O$, and $\odot Q$ does not enclose $O$.

*(a) Begin with our original problem.*

*(b) Invert with respect to an appropriate reference circle, yielding a new inverted problem.*

*(c) Solve the inverted problem.*

*(d) Invert again, then translate our result from the inverted setting back into our original one.*

Strategy 3.1.2 can be very useful, both in proving theoretical results like *Ptolemy's Theorem* (our topic for the session of November 9, 2024), as well as solving some problems from math competitions like the American Mathematics Competitions ("AMC") or the William Lowell Putnam Mathematical Competition ("the Putnam").

## 3.2 Exercises

3.2.1 First, we provide the following for background:

**Theorem 3.2.1** (Ptolemy's Theorem)**.** *Let $ABCD$ be a quadrilateral. Then if $ABCD$ is cyclic,*

$$AB \cdot CD + BC \cdot DA = AC \cdot BD. \tag{3.2.1}$$

*That is, in a cyclic quadrilateral, the sum of the product of the lengths of opposite sites is equal to the product of the lengths of the diagonals.*

See Figure 3.2.1 for reference.

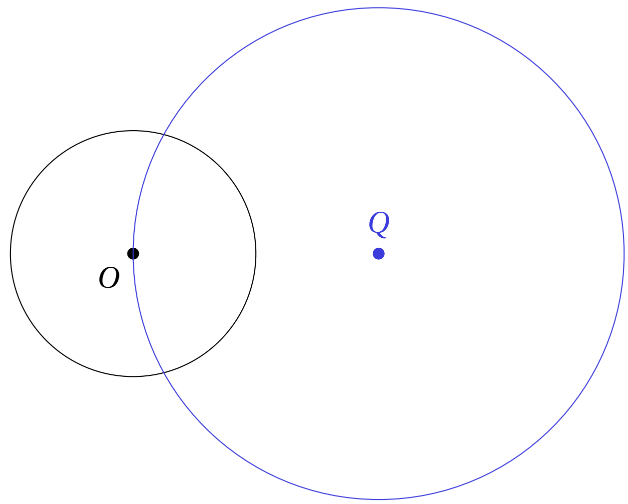



Figure 2.2.6: Computing the inverse of $\odot Q$ when $\odot Q$ passes through $O$. Shown: the case where $\odot Q$ also passes through two distinct points of $\odot O$.

**Theorem 3.2.2** (Ptolemy's Inequality)**.** *Let $ABCD$ be a quadrilateral. Then*

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD \tag{3.2.2}$$

*Further, equality in* (3.2.2) *holds if and only if $ABCD$ is a* cyclic *quadrilateral.*

Prove Ptolemy's Theorem and Inequality using inversion.

*Hints:* To apply inversion, one first needs to choose a suitable reference circle. Inverting with respect to the given circle will leave $A, B, C, D$ fixed, What might be your center?

In our original context, we are considering four points lying on a common circle. Our goal is to invert our original circle—inverting with respect to a *new* circle, to be clear!—to a line $\ell$, since working with lines it typically much simpler than working with circles.

Upon inverting, you should find three collinear points $A'$, $B'$, $C'$, falling in that order. On segment $\overline{A'C'}$, it follows that $A'B' + B'C' = A'C'$. Using Equation (2.2.1), manipulate $A'B' + B'C' = A'C'$ until you conclude Ptolemy's Theorem.

For Ptolemy's Inequality, note that if our quadrilateral is *not* cyclic, then the three points will *not* be collinear after inversion. For noncollinear points $A'$, $B'$, and $C'$, we will have $A'B' + B'C' > A'C'$, by the *triangle inequality*.

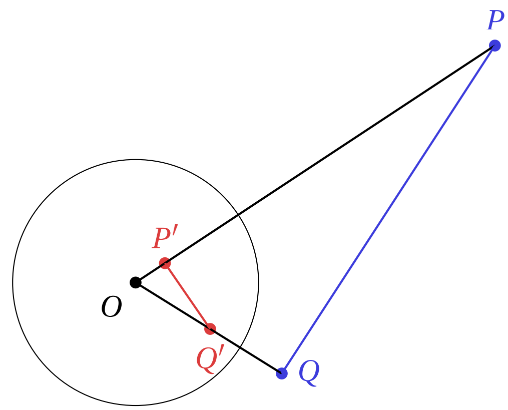



Figure 2.2.7: Computing $P'Q'$ in terms of $PQ$, $OP$, $OQ$, and $r$. Shown: the specific case where $P, Q$ both lie outside $\odot O$.

3.2.2 Let $\odot Q_1$, $\odot Q_2$, $\odot Q_3$, and $\odot Q_4$ be distinct circles such that $\odot Q_1$ is tangent to $\odot Q_2$ at $P_1$, $\odot Q_2$ is tangent to $\odot Q_3$ at $P_2$, $\odot Q_3$ is tangent to $\odot Q_4$ at $P_3$, and $\odot Q_4$ is tangent to $\odot Q_1$ at $P_4$, as in Figure 3.2.2. Prove that the points of tangency $P_1$, $P_2$, $P_3$, and $P_4$ lie on some common circle $\odot Q$.

*Hint:* It suffices to show that after inverting with respect to a well-chosen circle $\odot O$, the $Q_i$ lie on a common line. After inverting back, we can conclude that the original points $Q_i$ lie on some common circle.

*Remark.* This appears to be the problem being explored in [27].

3.2.3 In past sessions on November 18, 2023 and December 2, 2023, we explored numbers and points that are constructible with straightedge and compass. Given $P \neq O$ and our reference circle $\odot O$, explain how to construct using straightedge and compass the inverse $P'$ of $P$. Consider separately the cases where $P$ lies on $\odot O$, $P$ lies inside $\odot O$, and $P$ lies outside $\odot O$, respectively.

*Remark.* Many authors use these constructions to *define* inversion; see, for example, [36] (and others by the same author), among others.

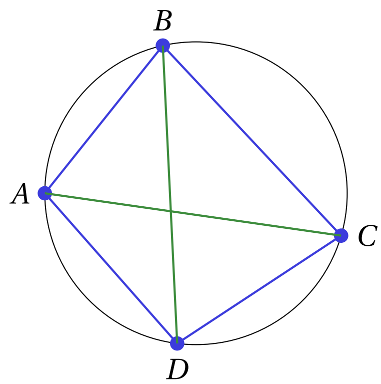



Figure 3.2.1: A cyclic quadrilateral $ABCD$ and its diagonals $\overline{AC}$ and $\overline{BD}$.

3.2.4 Let $\odot P_1$, $\odot P_2$, $\odot P_3$, and $\odot P_4$ be "kissing circles", in the sense that each $\odot P_i$ is tangent to the other $\odot P_j$. If each such circle $\odot P_i$ has radius $r_i$, then prove *Descartes' Theorem*:

$$\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)^2 = 2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}\right). \tag{3.2.3}$$

One approach using inversive geometry to prove Descartes' Theorem is found in [8].

*Remark.* In general, if three mutually tangent circles are given, Descartes' Theorem suggests there are two possible radii for a fourth circle to be mutually tangent. This ambiguity is because we may have either internal or external tangency. The typical convention is to allow for negative radii to convey whether a circle is bending towards or away from its neighbor.

Further, if we extend to generalized circles, including straight lines, then (3.2.3) can be reformulated in terms of the *curvatures* of these generalized circles. Though a straight line has no radius, it has zero curvature.

3.2.5 Let $\mathbb{C}$ denote the complex plane. Set $O = 0 = 0 + 0i$, and let $r = 1$, so that $\odot O$ is the circle in $\mathbb{C}$ of radius 1 and center at the origin. Prove that *relative to this particular circle* $\odot O$, the inverse of the point $z \neq 0$ is given by the formula

$$z' = \frac{1}{\overline{z}}, \tag{3.2.4}$$

where $\overline{z}$ denotes the *conjugate* of the complex number $z$.

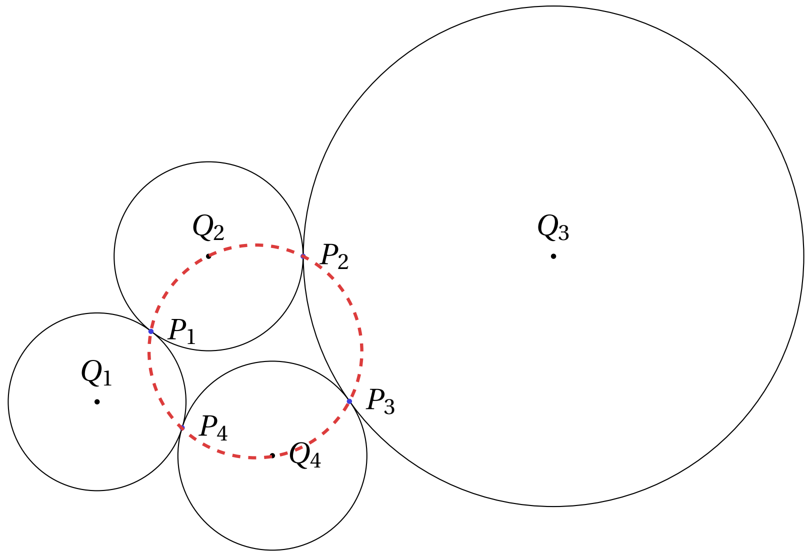



Figure 3.2.2: The points of tangency $P_i$ of tangency for these mutually tangent circles $\odot Q_i$ are themselves concyclic.

*Remark.* This can be seen as a refinement of the motivating example from Proposition 1.1.1, where we considered only reciprocation rather than reciprocating the conjugate.

# 4 Proving Properties of Inversion

## 4.1 Discussion

## 4.2 Exercises

4.2.1 Let $\Omega_1, \Omega_2$ be generalized circles, and let $P \neq O$ be a point lying on each $\Omega_i$. Prove that if these are inverted with respect to our reference circle $\odot O$, then $P'$ is a point of intersection of the inverses $\Omega_1'$ and $\Omega_2'$.

*Followup:* What happens if $P = O$ is such a point of intersection?

4.2.2 Consider any of the conjectures arising from the exercises in Section 2.2 about how the inverse of a generalized circle is a generalized circle. Prove at least one of these statements.

*Hint:* To prove, for example, the result from Exercise #2.2.1, first select the points $N$ and $F$ on $Q$ that are nearest and farthest, respectively, from the center $O$ of the reference circle. Because $\odot Q$ is a circle, any point $A$ on $\odot Q$ will be such that $\angle FAN$ is a right angle. What happens upon inverting $F$, $A$, and $N$ to $F'$, $A'$, and $N'$, respectively?

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