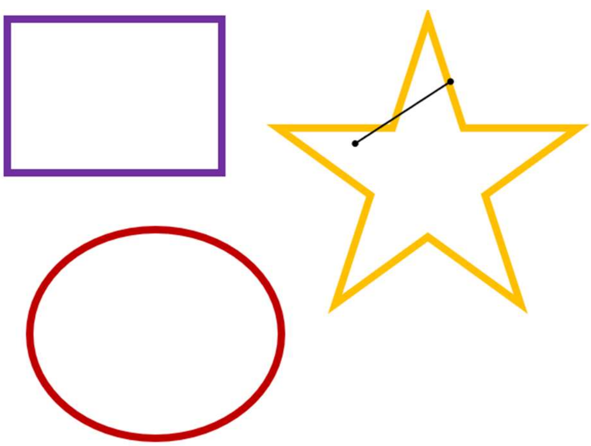
Chapel Hill Math Circle
Session 10 – February 15, 2025:
Spaghetti and Pizza[1] Numbers Part I
Beginners' Group (grades 1-3),
10:30-11:30a

Mr. Barman –
dilip@trianglemathinstitute.com

Supplies needed per student: 3-5 white paper plates, at least 6-8" in diameter (alternatively several circles drawn on paper), straightedge (like a ruler), several pieces of dry spaghetti or other long pasta, as well as paper and pencil

Welcome to Math Circle session 10. Let's talk about spaghetti, pizza, and uttapam – uttapam is a slightly sour, savory, thick Indian "pancake". Let's start with understanding convex & concave shapes.

Convex* and *concave shapes



A simple closed shape doesn't have any crossings, is flat, and has an "inside" and an "outside".Pick any two points inside a simple closed shape. If you can always connect them with a straight line segment that stays inside the shape, the shape is said to be *convex*. Otherwise the shape is *concave*. Think of a bear hiding in a nook or cranny of a concave shape. Normally we start working with simple closed convex shapes.

In the star we can easily find a pair of points such that when we connect them, we leave the star's boundary. That's not true with the rectangle and oval here. We say the rectangle and oval are convex and the star is concave.

Use green to draw a set of convex shapes and red to draw a set of concave shapes. Then use a straightedge or just draw the best you can to make a straight line segment through each of the shapes; use black. The goal is to divide the shape into as many pieces as possible. Try it!

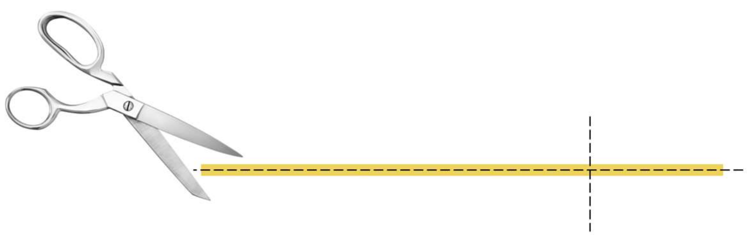
[1] To the right is a picture of a vegan pesto pizza that I made on April 25, 2006.

Did you find that you can always divide a convex shape into two pieces but a concave shape can sometimes have more than two pieces? We'll come back to this in a moment but first let's play with spaghetti.

Spaghetti Numbers

If you have some dry uncooked spaghetti or other straight pasta, we'll use it. You can also get by by drawing lines.

In real life pasta[2] has a width and in theory at least you could cut it in two ways, along the length or across the length:



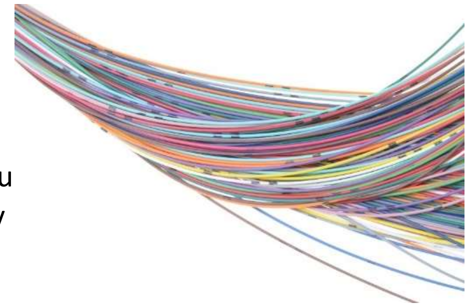
In reality you could cut it at different angles too but let's just consider these cases. Let's assume that the pasta is so thin as to be like string and that it can't be split the long way but just along its length like the vertical line above represents.

Please take a minute to investigate how many pieces you can cut a piece of spaghetti into.

- Use a piece of dry pasta and snap it with your fingers where you would have it cut
- If you don't have pasta, draw a piece of spaghetti and use a different color to represent where you can cut it
- I think that we will agree that before we start when we have 0 cuts there is 1 piece
- Let's try to cut the spaghetti into as many pieces as possible without picking up and moving the cut pieces
- How many pieces of spaghetti can we have after 1 cut? 2 cuts? 3 cuts? What do you think you would get after 500 cuts?

[2] The image of pasta here is in the public domain and created by Popo le Chien on Dec. 4, 2017. It was shared with a Creative Commons CC0 1.0 Universal Public Domain license and I downloaded it on Jan. 28, 2025 from commons.wikimedia.org/wiki/File:Linguine2.jpg.

Alternative spaghetti cutting



What happens if we relax our assumption that we can't move the string/pasta in between cuts? If you can move, how many pieces can you get with 2 cuts? 3 cuts? 4 cuts? Remember, we are trying to get as many pieces as possible.

Let's consider pizza now!

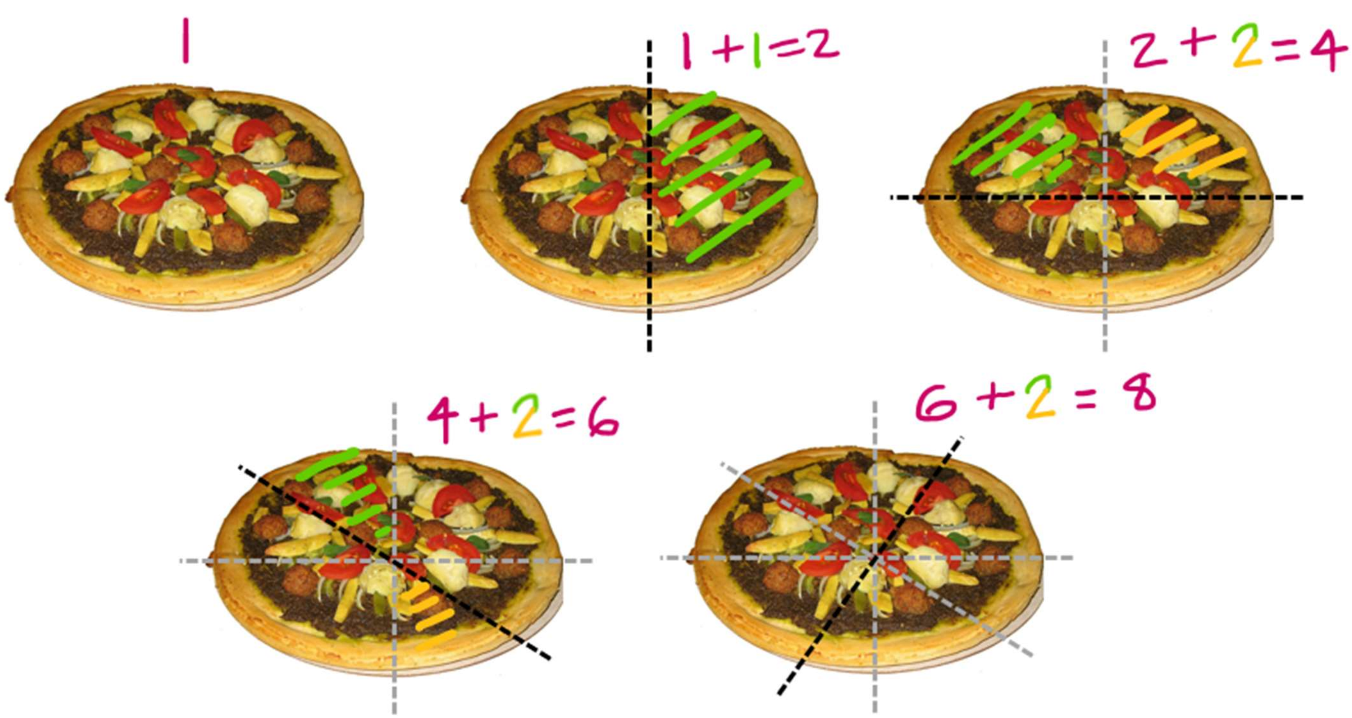
You should have some paper plates. If not you can make (rough is okay) circles on paper; we'll take these circles to be pizzas and later you're welcome to color them to make them look like pizzas if you want.

How many pieces can you cut the pizza into with 1 straight cut? 2 straight cuts? 3? How about 20? For today let's assume that you are not picking up the slices after cutting but leaving the pizza alone. By the way do you know what uttapam is?

What did you come up with? How many pieces can you get with 0 slices? 1 slice? 2, 3, and 4 slices? More?

Let's think about the traditional way of slicing pizza into wedges and call these Modified Pizza Numbers. Here we try to make as many pieces as possible but with the constraint that we cut a pizza the traditional way into wedges. The obvious cases are 0 cuts (1 piece) and 1 cut (2 pieces).

Here is the pizza that I made on April 25, 2006. If we cut it the traditional way into wedges, after the first cut, each new cut adds 2 new pieces, doesn't it?



But can you do better? Believe it or not, I can make even more slices with straight cuts as this table shows. Can you figure out how to do this?

Cuts	0	1	2	3	4	5	6
Pieces	1	2	4	7	11	16	22

Can you make 7 pieces with just 3 cuts? 11 pieces with just 4 cuts? Try it! Our goal is the most pieces – we don't care if the pieces are different sizes, have crust or not, or are of strange shapes.

Please keep playing with this idea. How many pieces of pizza would you get if you were allowed to rearrange the pieces in between slices? How does this compare to Spaghetti Numbers where you can rearrange the pieces? What patterns do you see?

There's a lot more to do! Next time we will quickly review what we did this time and have part 2 of this topic and even talk about Watermelon Numbers! We'll also look at this picture below (and/or similar pictures); this one is from my 2nd grade student Jayden in the Stanford Math Circle from February 2025.

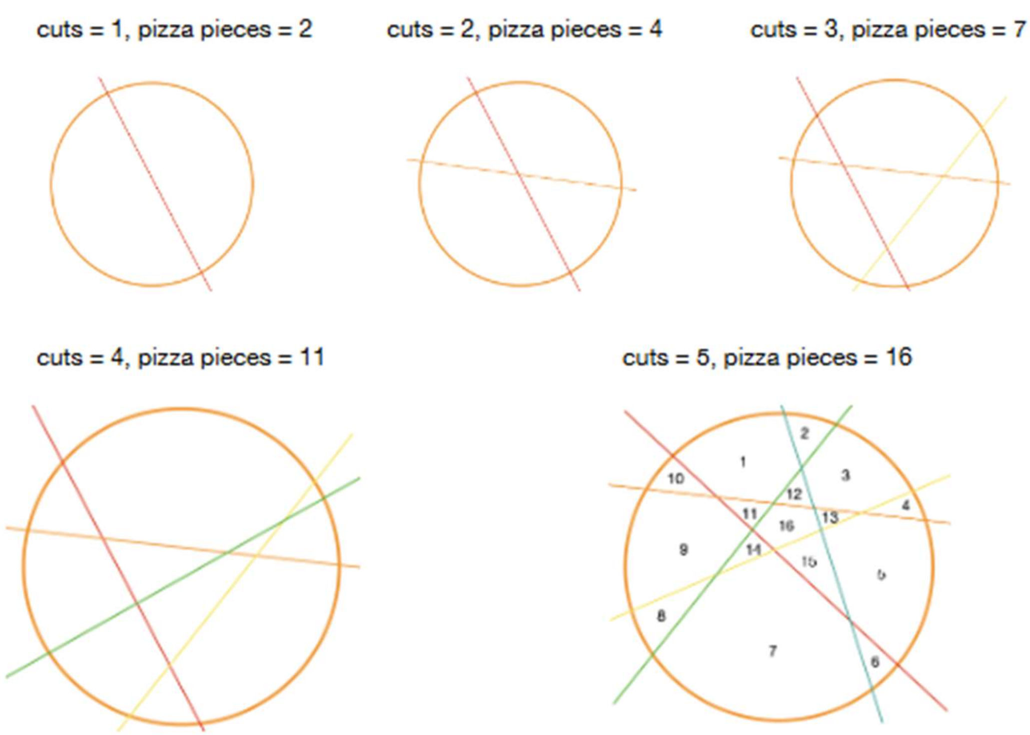
Have Fun!
Mr. Barman

Notes for Parents[3]

I’m excited about presenting this topic and hope to have had a surprise at the end with introducing students (and parents) to traditional South Indian uttapams, gluten-free and vegan savory “pancakes”. They are easier to cut than pizza. Will we be able to “make the cut” and reach the theoretical maximum number of pieces?

There’s lots of interesting discussion that we could have here – or that you could with your child. It’s a great time to discuss theory vs. practice. If you get on a flight that is scheduled to leave RDU airport at 12:08p, make a 40-minute connection, and arrive at a destination at 3:32p, say, how likely is it that this actually happens? How is theory different from practice?

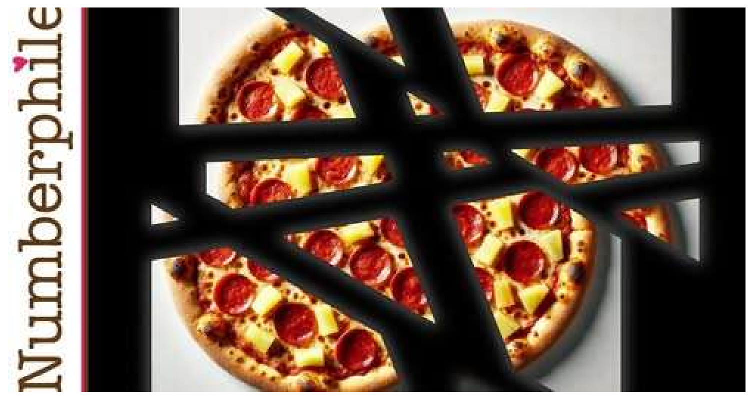
Cutting with rearranging is straightforward in both pizza and spaghetti cases; it’s simply doubling. So the theoretical maximum number of pieces with $n \in \mathbb{Z}^+$ (i.e., n is an element or comes from the set of positive integers) cuts of a line or a convex closed object is 2^n. But we can’t get there without rearranging, a more interesting and challenging case.



Clearly the 0th “Pizza Number” (pieces with 0 cuts) is 1 and the 1st is 2. It’s also clear that with 2 cuts we get maximally 4 pieces. We will continue next time to explore these numbers. These diagrams to the left by my Stanford 2nd grade student Summer from February 2025 shows that we have to be judicious with our cuts.

[3] The pizza picture here is from February 2025 by 2nd grader Evan, one of my Stanford Math Circle students.

I highly recommend a Numberphile[4] video for you to see but please don't share yet with your child this or any of these other "spoilers". Their video *The Lazy Way to Cut Pizza*[5] by math teacher Tom Crawford[6], featured in the video, tries to make the 22 pieces that we should be able to get with 6 straight cuts (T_6+1 = 1+2+3+4+5+6 + 1 = 21+1 = 22; do you see the connection to Triangular Numbers?). By the way, he only gets 18 – I will likely show this next time and see if students can guess why the practice didn't meet the theory.



I wouldn't expect students to follow the math but there is a related topic called monohedral disc tiling. The discs look like the ones to the right[7]. I encourage you to see a nice Vsauce[8] video *The Pizza Theorem*[9] and a shorter one by Maths Doctor[10] called *How do Mathematicians Cut Pizza? SOLUTION!*[11] (there's a shorter introduction video but really it's just the question asked at the start).

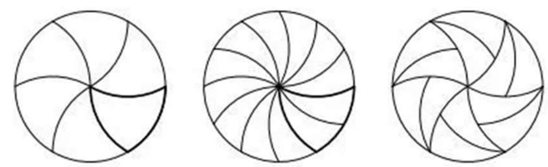
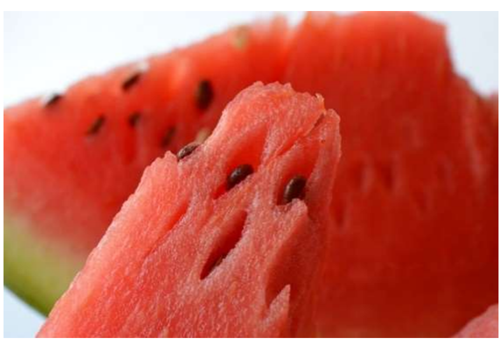


Figure 4: A disk tiled by 6 shields, and two tilings with 12 tiles (T^2_{12}).

Figure 13: Tilings T^1_{20}, T^1_{28}, T^1_{36}.



I encourage you to try to eat pizza or some sort of pie this week and help your child find the Pizza Numbers. If you have time, try Watermelon Numbers by cutting a three-dimensional object like watermelon – or cake, as the actual name is Cake Numbers. You should find that the sequence starts with 1, 2, 4, 8, but then goes on to 15 (not 16), 26, and 42. It is OEIS sequence A000125[12].

Till next time!

Dilip

Dilip Barman dilip@trianglemathinstitute.com

[4] numberphile.com
[5] youtube.com/watch?v=Xd9UZSodeN8
[6] tomrocksmaths.com
[7] These are from a short and very readable popular article "The complex yet revolutionary way to cut your pizza into equal slices" by Leah Cohen on yahoo!lifestyle; the photo credit is Joel Haddley and Stephen Worsley who came up with this curved tiling; I got the picture on Feb. 4, 2025 from au.lifestyle.yahoo.com/the-new-way-to-cut-your-pizza-into-equal-slices-30560964.html. You can read their research paper online at arxiv.org/pdf/1512.03794v1.
[8] Vsauce.com
[9] youtube.com/watch?v=SXgF57NWJgs
[10] youtube.com/channel/UCI_vLyzRi06rIikgw-ATM5Q
[11] youtube.com/watch?v=HjcxXnRg1is
[12] oeis.org/A000125