Ptolemy's Theorem

Abstract

Ptolemy's Theorem is a result relating the lengths of the sides and diagonals in a *cyclic quadrilateral*, meaning a quadrilateral all of whose vertices lie on a common circle $\odot O$. In this session, our goal is to understand what Ptolemy's Theorem says, how to prove it, and how to apply it. In particular, Ptolemy's Theorem has the *Pythagorean Theorem* as a corollary.

Background needed: Prerequisites include some basic results from geometry, especially involving *inscribed angles* (and *enclosed* or *intercepted arcs*) within a circle and the *inscribed angle theorem*. Some results will use trigonometry, though there will be enough to explore even for students who have yet to study this.

Contents

0 Warmup

As prerequisites for this session, it will help to answer the following first. You do *not* need to know these answers already, and many questions will be revisited later in the worksheet.
Exercises:

0.1 What are *similar polygons*? In particular, if $\triangle ABC$ is similar to $\triangle DEF$ (in that order!), denoted $\triangle ABC \sim \triangle DEF$, what does this mean? Conversely, if we want to prove that $\triangle ABC \sim \triangle DEF$, how might we do so?

0.2 What is a *cyclic polygon*? Can you provide some examples?

0.3 In a circle, what is an *inscribed angle*? What is a *central angle*? What is an *intercepted arc*? Can you give some examples?

0.4 What is the *inscribed angle theorem* for circles? In a circle with center $\odot O$, how does the measure of an inscribed angle $\angle ABC$ to an arc $\overset{\frown}{AC}$ compare to the measure of the central angle $\angle AOC$?

0.5 In a circle $\odot O$ with points A, B lying on the circle, what is an arc $\overset{\frown}{AB}$? For A, B on $\odot O$, how does one measure $\overset{\frown}{AB}$?

0.6 What is an *isosceles triangle*? If $\triangle ABC$ is an isosceles triangle with $\overline{AB} \cong \overline{BC}$, what can we say about angles $\angle BAC$ and $\angle BCA$?

0.7 Let $n \geq 3$ be a positive integer. If $\mathcal{P}$ is a convex polygon with n vertices (and likewise n edges), what is the sum of the *internal angles* (or *interior angles*) of $\mathcal{P}$? What is the sum of the *external angles* (or *exterior angles*) of $\mathcal{P}$

0.8 If θ is an angle, what are the *sine* and *cosine* of θ, denoted $\sin\theta$ and $\cos\theta$, respectively?

For example, say that $\triangle ABC$ is a right triangle with $\overline{AB} \perp \overline{BC}$, as in Figure 0.1. Relative to the side lengths $|AB|$, $|BC|$, and $|AC|$, how do we compute $\sin\angle A$ and $\cos\angle A$?

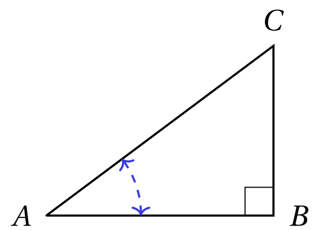


Figure 0.1: In right triangle $\triangle ABC$, how does one compute $\sin\angle A$ and $\cos\angle A$?

1 Circles, Angles, and Cyclic Quadrilaterals

1.1 Discussion

We begin by introducing some terminology:

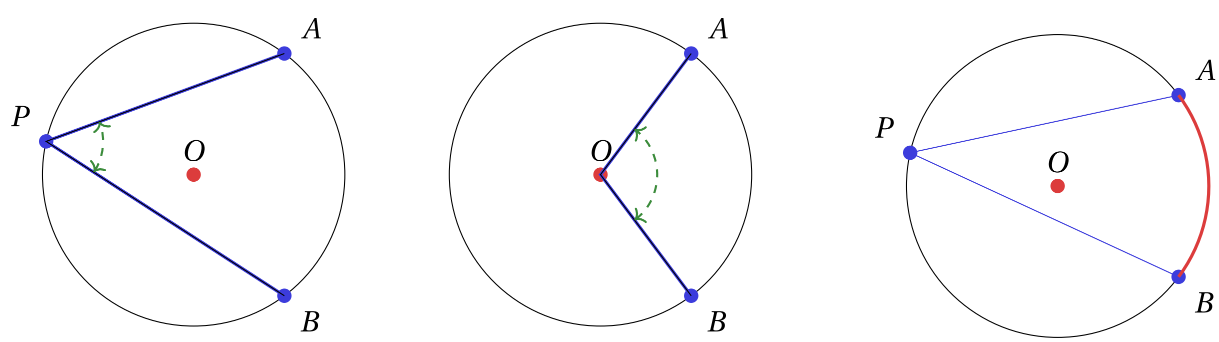
Definition 1.1. Let O be a point, and let $\odot O$ denote a circle whose center is O.

(a) An *inscribed angle* for arc $\overparen{AB}$ is the angle formed in the interior of a circle where two chords intersect at a unique point P.

(b) The *central angle* for arc $\overparen{AB}$ is an angle whose vertex is O, the center of the circle.

(c) The *intercepted arc* (or *subtended arc*) for $\angle APB$ is the arc $\overparen{AB}$ on $\odot O$ and opposite[1] P.

See Figure 1.1 for details.



(a) Inscribed angle $\angle APB$ in $\odot O$.

(b) Central angle $\angle AOB$ in $\odot O$.

(c) Intercepted arc $\overparen{AB}$ by inscribed $\angle APB$.

Figure 1.1: An inscribed angle, a central angle, and an intercepted arc.

[1] If P lies on $\overline{AB}$, so that A, P, and B are colinear, then there may be ambiguity which of the two congruent arcs is intended. For our purposes, the intended arc shall be selected explicitly.

Definition 1.2. Let $ABCD$ be a quadrilateral with vertices A, B, C, and D. Then $ABCD$ is a *cyclic quadrilateral* if and only if there is a circle $\odot O$ with center O such that each vertex of $ABCD$ lies on $\odot O$.

More generally, if $A_1, A_2, \ldots, A_n$ are distinct points in the plane, then the polygon $A_1 A_2 \ldots A_n$ is a *cyclic polygon* if and only if there is a circle $\odot O$ with center O such that each vertex A_i lies on $\odot O$.

Cyclic quadrilaterals will be our primary focus and the setting of Ptolemy's Theorem. For reference on cyclic quadrilaterals, see Figure 1.2.

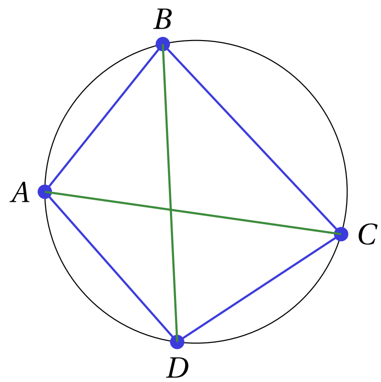


Figure 1.2: A cyclic quadrilateral $ABCD$ and its diagonals $\overline{AC}$ and $\overline{BD}$.

The following is an important theorem relating inscribed to central angles for a common arc $\overset{\frown}{AC}$.

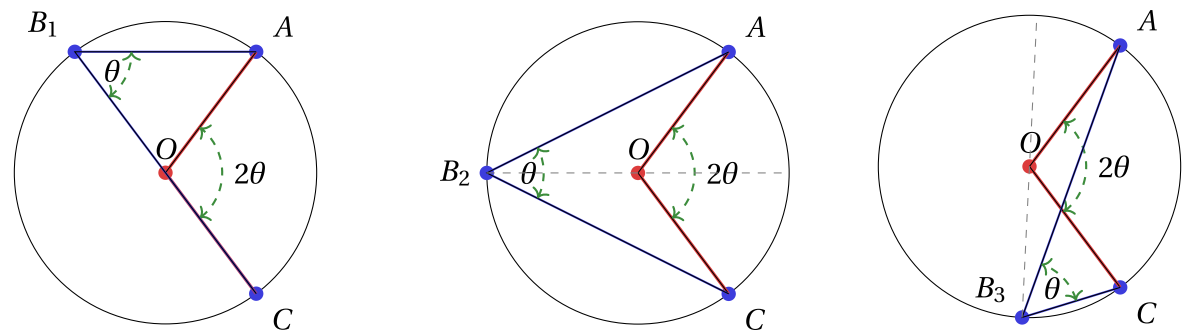
Theorem 1.3 (Inscribed Angle Theorem)**.** *Let $\odot O$ be a circle with center O. If A, C lie on $\odot O$, and B is another point on $\odot O$ with B not lying on arc $\overset{\frown}{AC}$, then $m\angle ABC = \frac{1}{2} m\angle AOC$.*

In particular, independent *of the specific position of B, every angle $\angle ABC$ that inscribes $\overset{\frown}{AC}$ has the* same *measure: $\frac{1}{2} m\angle AOC$.*

Figure 1.3 illustrates the primary examples for the Inscribed Angle Theorem. Namely, either B lies where line $\overleftrightarrow{OC}$ meets $\odot O$ (as in Figure 1.3a), $\overleftrightarrow{OB}$ intersects $\overset{\frown}{AC}$ (as in Figure 1.3b), or $\overleftrightarrow{OB}$ does *not* intersect $\overset{\frown}{AC}$ (as in Figure 1.3c). In either of these cases, if $m\angle AOC = 2\theta$, then $m\angle ABC = \theta$.

1.2 Exercises

1.2.1 Consider the following cyclic quadrilaterals, with side and diagonal lengths as indicated in Figure 1.4. For each example, compute $|AB| \cdot |CD| + |BC| \cdot |DA|$ and $|AC| \cdot |BD|$. (Please feel free to use a calculator, too!)

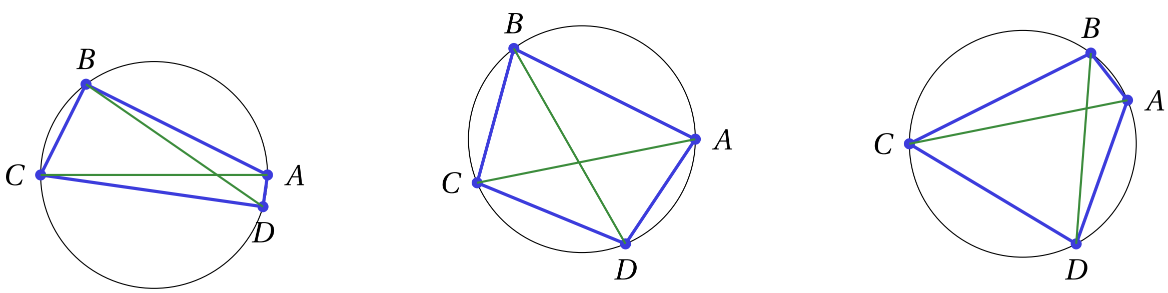


(a) B_1 lies on $\overleftrightarrow{OC}$ and $\odot O$.

(b) $\overleftrightarrow{OB_2}$ intersects $\widehat{AC}$.

(c) $\overleftrightarrow{OB_3} \cap \widehat{AC} = \varnothing$.

Figure 1.3: The three instances of Theorem 1.3, the Inscribed Angle Theorem. Note that the three angles $\angle AB_1C$, $\angle AB_2C$, and $\angle AB_3C$ have measure $\frac{1}{2}m\angle AOC$.



(a) Exercise #1.2.1(a).

(b) Exercise #1.2.1(b).

(c) Exercise #1.2.1(c).

Figure 1.4: In cyclic quadrilaterals $ABCD$, compute $|AB|\cdot|CD|+|BC|\cdot|DA|$ and $|AC|\cdot|BD|$.

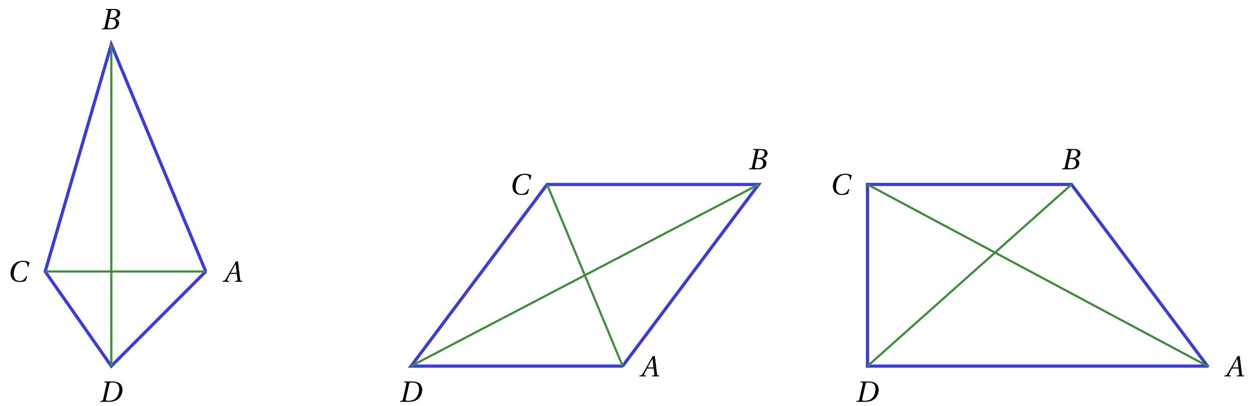
(a) $|AB| = 20\sqrt{5}$, $|BC| = 10\sqrt{5}$, $|CD| = 35\sqrt{2}$, $|DA| = 5\sqrt{2}$, $|AC| = 50$, $|BD| = 15\sqrt{10}$.

(b) $|AB| = 52\sqrt{5}$, $|BC| = 7\sqrt{130}$, $|CD| = 65\sqrt{2}$, $|DA| = 20\sqrt{13}$, $|AC| = 25\sqrt{26}$, $|BD| = 16\sqrt{65}$.

(c) $|AB| = 51\sqrt{130}$, $|BC| = 884\sqrt{5}$, $|CD| = 325\sqrt{34}$, $|DA| = 100\sqrt{221}$, $|AC| = 425\sqrt{26}$, $|BD| = 143\sqrt{170}$.

(d) **Challenge:** How do you think one could produce the examples in Exercises #1.2.1(a)–1.2.1(c)? Specifically, how could one verify that each $ABCD$ is cyclic, and how to compute the lengths of each side and diagonal? Can describe a method to create your own verifiably cyclic quadrilaterals?

1.2.2 In the following *noncyclic* quadrilaterals, we again compute the values $|AB| \cdot |CD| + |BC| \cdot |DA|$ and $|AC| \cdot |BD|$.



(a) Exercise #1.2.2(a). (b) Exercise #1.2.2(b). (c) Exercise #1.2.2(c).

Figure 1.5: In *noncyclic* quadrilaterals $ABCD$, compute $|AB| \cdot |CD| + |BC| \cdot |DA|$ and $|AC| \cdot |BD|$.

(a) $|AB| = 26$, $|BC| = 25$, $|CD| = \sqrt{149}$, $|DA| = 10\sqrt{2}$, $|AC| = 17$, and $|BD| = 34$.

(b) $|AB| = 15$, $|BC| = 14$, $|CD| = 15$, $|CD| = 14$, $|DA| = 15$, $|AC| = 13$, and $|BD| = \sqrt{673}$.

(c) $|AB| = 10$, $|BC| = 9$, $|CD| = 15$, $|CD| = 8$, $|DA| = 8$, $|AC| = 17$, and $|BD| = \sqrt{145}$.

1.2.3 Combining your observations from Exercises #1.2.1–1.2.2, can you formulate any conjectures?

1.2.4 Prove Theorem 1.3, the Inscribed Angle Theorem.

1.2.5 Let $ABCD$ be a cyclic quadrilateral, as in Figure 1.2. Prove that $m\angle A + m\angle C = m\angle B + m\angle D = 180°$. That is, in a cyclic quadrilateral, opposite angles are *supplementary*. As a corollary, if any pair of opposite angles in a quadrilateral is supplementary, then the quadrilateral is cyclic.

Challenge: Can you prove the converse? That is, can you prove that if $ABCD$ is a quadrilateral such that opposite angles are supplementary, then $ABCD$ is cyclic?

1.2.6 If we are given a circle $\odot O$, then it is easy to construct a quadrilateral whose vertices lie on $\odot O$. That is, given a circle, we know how to construct a cyclic quadrilateral.

If you are given some quadrilateral, how, in principle, can you determine whether it is cyclic? Obviously searching through all infinitely many possible circles is impossible.

Hint: Any triangle $\triangle ABC$ has a unique *circumcircle* $\odot O$, such that A, B, and C all lie on $\odot O$. How can we use this to help determine whether a given quadrilateral $ABCD$ is cyclic?

2 Ptolemy's Theorem: Statement and Consequences

2.1 Discussion

Motivated by our examples from Section 1, our goal is to first explore, and later prove, the following:

Theorem 2.1 (Ptolemy's Theorem)**.** *Let $ABCD$ be a quadrilateral. Then if $ABCD$ is cyclic,*

$$|AB| \cdot |CD| + |BC| \cdot |DA| = |AC| \cdot |BD|. \tag{2.1}$$

That is, in a cyclic quadrilateral, the sum of the product of the lengths of opposite sites is equal to the product of the lengths of the diagonals.

Further, the converse to Ptolemy's Theorem is also true:

Theorem 2.2 (Converse to Ptolemy's Theorem)**.** *Let $ABCD$ be a quadrilateral such that* (2.1) *holds. Then $ABCD$ is cyclic.*

Combining Ptolemy's Theorem with its converse, this means that $ABCD$ is cyclic *if and only if* (2.1) holds. The following theorem synthesizes Theorems 2.1–2.2, generalizing the computations we made in Exercises #1.2.1–1.2.2:

Theorem 2.3 (Ptolemy's Inequality)**.** *Let $ABCD$ be a quadrilateral. Then*

$$|AB| \cdot |CD| + |BC| \cdot |DA| \geq |AC| \cdot |BD| \tag{2.2}$$

Further, equality in (2.2) *holds if and only if $ABCD$ is a* cyclic *quadrilateral.*

Put differently, synthesizing Theorems 2.1, 2.2, and 2.3, we have the following: let $ABCD$ be a quadrilateral. Then

$|AB| \cdot |CD| + |BC| \cdot |DA| = |AC| \cdot |BD|$ if and only if $ABCD$ is cyclic, and
$|AB| \cdot |CD| + |BC| \cdot |DA| > |AC| \cdot |BD|$ if and only if $ABCD$ is *not* cyclic.

In the exercises for Subsection 4.2 later, our primary goal will be to prove Theorem 2.1, then leave as open questions how to prove Theorems 2.2–2.3. In Subsection 2.2 immediately below, we explore consequences of Ptolemy's Theorem, its converse, and Ptolemy's Inequality.

2.2 Exercises

2.2.1 Prove that the *Pythagorean Theorem* is a corollary to Ptolemy's Theorem. That is, prove that if $\triangle ABC$ is a right triangle with right angle at B, $a := |BC|$, $b := |AC|$, and $c := |AB|$, then $a^2 + b^2 = c^2$.

Hint: If $\triangle ABC$ is a right triangle with right angle at C, what happens upon forming the *circumcircle* for $\triangle ABC$? From this, how might you form a cyclic quadrilateral relevant to using Ptolemy's Theorem to prove the Pythagorean Theorem? Draw a picture as in Figure 2.1!

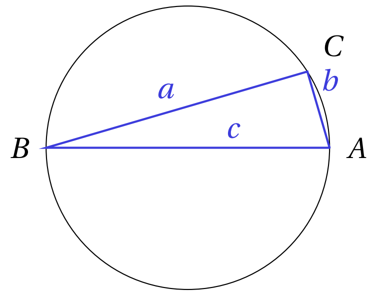


Figure 2.1: Showing Ptolemy's Theorem implies the Pythagorean Theorem for right triangle $\triangle ACB$.

2.2.2 Let $\odot O$ be a circle of *diameter* 1 (and thus of *radius* $\frac{1}{2}$). Prove that if $\overline{AB}$ is a chord in $\odot O$ formed by a central angle having measure 2θ, then $|AB| = \sin\theta$.

Hint: Form the angle bisector of $\angle AOB$, and let M be where the bisector meets $\overline{AB}$. Using Exercise #0.8 from the Warmup, can you compute $|AM|$? From that, can you compute $|AB|$?

2.2.3 Let α, β be measure of angles such that $0 \le \alpha < 180°$, $0 \le \beta < 180°$, and $\alpha + \beta < 180°$. Prove that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta. \tag{2.3}$$

Hint: Review the diagram in Figure 2.2, where $\odot O$ has diameter 1. In the context of Exercise #2.2.2, what can you conclude?

Note: The proof of this identity using Ptolemy's Theorem appears in [8] and [3], as well as in many other sources. There are similar Ptolemy's Theorem-based proofs for trigonometric identities such as $\sin(\alpha - \beta)$ and $\cos(\alpha + \beta)$, too.

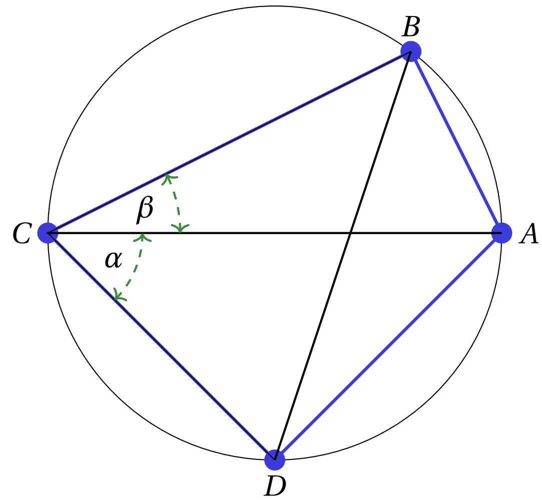


Figure 2.2: Proving $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\beta\sin\alpha$ via Ptolemy's Theorem in a circle of diameter 1.

3 Ptolemy's Theorem in Competition Math and Related Problems

3.1 Discussion

Ptolemy's Theorem has applicability to a number of problems, including some from mathematics competitions. The following surveys many examples, especially those compiled by [6], [19], and especially *Art of Problem Solving (AoPS)* in [1]. Some of these exercises are quite challenging, particularly—but not exclusively—those taken from the International Mathematical Olympiad (IMO). (The early exercises, though, have been chosen to be more accessible.)

3.2 Exercises

3.2.1 Equilateral Triangle Identity (from [1]): "Let $\triangle ABC$ be an equilateral triangle. Let P be a point on minor arc AB of its circumcircle. Prove that $PC = PA + PB$."

For reference, see Figure 3.1. For a video demostration of this identity, see [7].

3.2.2 Problem 4.2 in[19]: "Point P is chosen on the arc CD of the circumcircle of a square

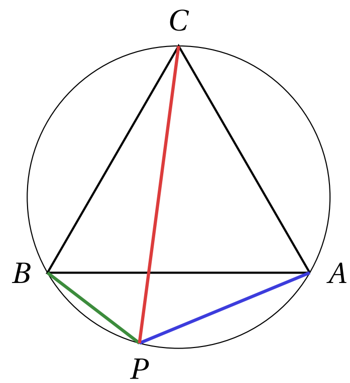


Figure 3.1: If $\triangle ABC$ is equilateral, then $PC = PA + PB$.

$ABCD$. Prove that

$$|PA| + |PC| = \sqrt{2} \cdot |PB|."$$

3.2.3 Regular Heptagon Identity (from [1]): "In a regular heptagon $ABCDEFG$, prove that: $\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AE}$."

Note: This identity also appears in [8].

3.2.4 **2004 American Mathematics Competitions (AMC) 10B:** "In triangle ABC we have $AB = 7$, $AC = 8$, $BC = 9$. Point D is on the circumscribed circle of the triangle so that AD bisects angle BAC. What is the value of AD/CD?

(A) $\frac{9}{8}$ (B) $\frac{5}{3}$ (C) 2 (D) $\frac{17}{7}$ (E) $\frac{5}{2}$"

3.2.5 Problem 4.4 in [19]: "The angle bisector of $\angle BAC$ of triangle ABC meets its circumcircle at D. Prove that

$$|AB| + |AC| \leq 2 \cdot |AD|."$$

3.2.6 **2023 American Invitational Mathematics Examination (AIME) I, Problem 5:** "Square $ABCD$ is inscribed in a circle. Point P is on this circle such that $AP \cdot CP = 56$, and $BP \cdot DP = 90$. What is the area of the square?"

3.2.7 **1991 AIME Problem 14:** "A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A."

3.2.8 Cyclic Hexagon (from [1]): "A hexagon with sides of lengths 2, 2, 7, 7, 11, and 11 is inscribed in a circle. Find the diameter of the circle."

Note: From context, I believe the statement of this problem suggests either that you may place the sides in any order you prefer and answer that instance, or that your answer is independent of how you make such an assignment. To simplify matters, assign half of each pair to a single hemisphere of a circle, and the other half to the opposite semicircle.

3.2.9 **2016 AMC 10A Problem 24 and 2016 AMC 12 Problem 21:** "A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?

(A) 200 **(B)** $200\sqrt{2}$ **(C)** $200\sqrt{3}$ **(D)** $300\sqrt{2}$ **(E)** 500"

3.2.10 Problem 4.5 in [19]: "Let $ABCD$ be a cyclic quadrilateral with $\angle CBA = \angle ADC = 90°$. Prove that

$$|BD| = |AC| \cdot \sin \angle BAD."$$

Note: An equivalent version of this appears as Corollary 2 in [6].

3.2.11 **2016 AIME I Problem 6:** "In $\triangle ABC$ let I be the center of the inscribed circle, and let the bisector of $\angle ACB$ intersect AB at L. The line through C and L intersects the circumscribed circle of $\triangle ABC$ at the two points C and D. If $LI = 2$ and $LD = 3$, then $IC = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$."

3.2.12 **2001 Baltic Way #7:** "A parallelogram $ABCD$ is given. A circle passing through A meets the line segments AB, AC and AD at inner points M, K, N, respectively. Prove that

$$|AB| \cdot |AM| + |AD| \cdot |AN| = |AK| \cdot |AC|."$$

Note: This also appears as Problem 4.5 in [19].

3.2.13 **1995 International Mathematical Olympiad (IMO) Problem 5:** "Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \frac{\pi}{3}$ [radians, or 60°]. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = \frac{2\pi}{3}$ [radians, or 120°]. Prove that $AG + GB + GH + DH + HE \geq CF$."

3.2.14 **1996 IMO Problem 2:** "Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that AP, BD, CE meet at a point."

4 Proving Ptolemy's Theorem and Inequality

4.1 Discussion

Thus far, we have motivated Ptolemy's Theorem and its progeny. We have also deduced consequences of Ptolemy's Theorem, from trigonometric identities to the Pythagorean Theorem. We have yet to *prove* these theorems, though. In this section, we shall present a remarkably intuitive way of confirming that Ptolemy's Theorem holds. Proving the converse to Ptolemy's Theorem and Ptolemy's Inequality hold, though, will be left as open exercises.

There are, however, many, many different valid methods for proving Ptolemy's Theorem and its relatives. One interesting technique for proving Ptolemy's Theorem is *inversion* or *inversive geometry*. A complete introduction to inversion would merit its own session, but the following provide an introduction to inversion and its relevance to Ptolemy's Equation: [17], [16], [15] (itself in [18]), [10], and Proof 9 in [14].

In addition to inversion and what we shall present in Subsection 4.2 below, there are methods using complex numbers (as in [2] and Proof 7 in [14]), by similar triangles (Proofs 1–2 in [14], the proof in [19], by trigonometry (Proofs 4–6 in [14]), vectors (Proof 8 in [14]),

4.2 Exercises

4.2.1 The proof method outlined here for Ptolemy's Theorem, Theorem 2.1, is geometric. See, for example, [8], Proof 4 in [14], [9], [7], and [4], among others. (*Caution:* These sources often use different labeling for the respective vertices, so be careful to understand precisely which triangles are being selected.) Throughout, let $ABCD$ be a cyclic quadrilateral, as in Figure 4.1.

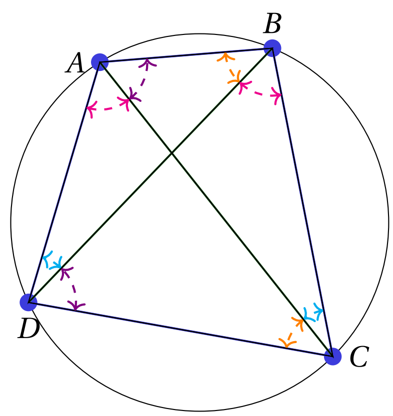


Figure 4.1: Proving Ptolemy's Theorem for cyclic quadrilateral $ABCD$.

We present this method over multiple steps:

(a) Show that $\angle ABD \cong \angle ACD$, $\angle BCA \cong \angle ADB$, $\angle BAC \cong \angle BDC$, and $\angle CBD \cong \angle CAD$.

(b) Prove that $m\angle BAC + m\angle ADB = m\angle ADC$, $m\angle ABD + m\angle CAD = m\angle ABC$, and $m\angle BCA + m\angle BAD + m\angle ACD = 180° = \pi$ radians.

(c) Scale $\triangle ABC$ by a factor of $|AD|$, $\triangle ADB$ by a factor of $|AC|$, and $\triangle ACD$ by a factor of $|AB|$. Take the rescaled triangles, and arrange as you see in Figures 4.2 and 4.3. Prove the figure formed is a parallelogram.

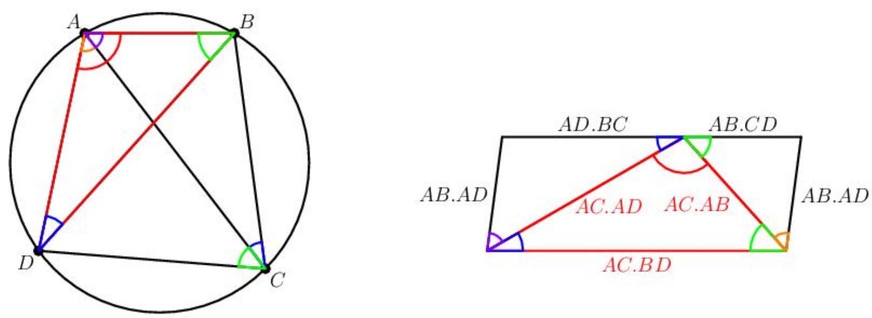


Figure 4.2: Visualizing our rescaled triangles, now arranged as a parallelogram. Image found in [14]. *Note:* Colors are different from those in Figure 4.1.

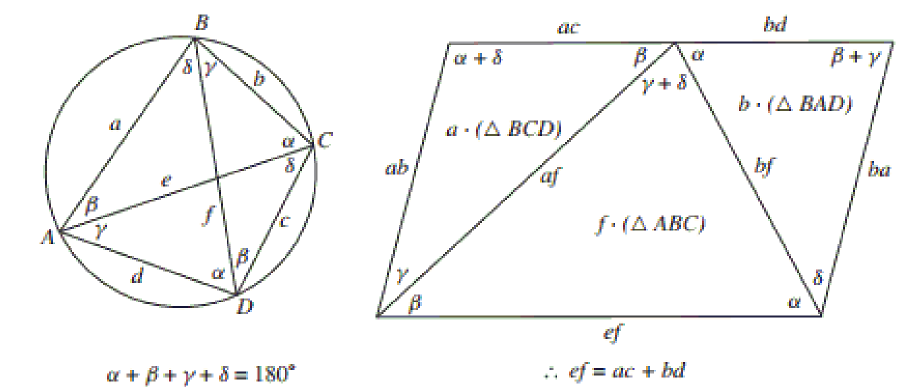


Figure 4.3: Visualizing our rescaled triangles, now arranged as a parallelogram. Image found in [14]. *Note:* The labeling the vertices is different from that in Figure 4.1.

(d) Use properties of parallelograms to conclude Ptolemy's Theorem:

$$|AB| \cdot |CD| + |BC| \cdot |DA| = |AC| \cdot |BD|.$$

(e) **Challenge:** Can you use the method above to show that for a general quadrilateral $ABCD$ that is *not* cyclic, then the process of trying to form a parallelogram fails? If so, can you deduce the converse to Ptolemy's Theorem, Theorem 2.2?

A short, clear animation that illustrates the ideas underlying this proof is available from [9].

4.2.2 The following will sketch steps for a proof of the converse to Ptolemy's Theorem, Theorem 2.2. Let $ABCD$ be a quadrilateral. We must show that if

$$|AB| \cdot |CD| + |BC| \cdot |DA| = |AC| \cdot |BD|,$$

then $ABCD$ is cyclic. Using the *contrapositive* of the statement in Theorem 2.2, we have the following, logically equivalent statement:

$$\textit{Goal: } \text{If quadrilateral } ABCD \text{ is } \textit{not} \text{ cyclic, then } |AB| \cdot |CD| + |BC| \cdot |DA| \neq |AC| \cdot |BD|. \tag{4.1}$$

(a) Let $\triangle XYZ$ be a triangle. Explain how for any such triangle, there exists a circle $\odot O$ such that X, Y, and Z all lie on $\odot O$.

That is, explain why *every* triangle is cyclic. (You may recall that this was presented above as a hint for Exercise #1.2.6. Again, such a $\odot O$ is called the *circumcircle* for $\triangle XYZ$.)

Remark. Those of you familiar with constructions with compass and ruler may recall how, specifically, to construct this circle. Specifically, the center O of the circumcircle for $\triangle XYZ$ will lie at the intersection of the perpendicular bisectors of $\overline{XY}$, $\overline{YZ}$, and $\overline{ZX}$.

(b) Show that $ABCD$ is *not* cyclic if and only if D does *not* lie on the circumcircle for $\triangle ABC$.

Note: Naturally, similar results hold when considering other triples among the vertices of $ABCD$. For example, $ABCD$ is not cyclic if and only if B does not lie on the circumcircle of $\triangle ACD$, with analogous results for C and A.

For a visual proof of Ptolemy's Inequality (Theorem 2.3) in the spirit of our proof method in Exercise #4.2.1, see [8]. (That video, along with other references, also discusses generalizations of Ptolemy's Theorem and Inequality to three dimensions.) See other methods for proving these theorems in the references.

References

[1] Art of Problem Solving. Ptolemy's theorem. https://artofproblemsolving.com/wiki/index.php/Ptolemy%27s_theorem. online: retrieved November 7, 2024.

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CHMC Exit Ticket

- What did you enjoy about today's topic?
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- Was there anything you thought was too easy?
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