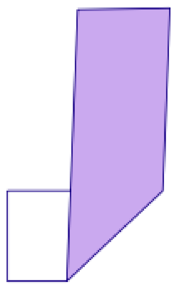
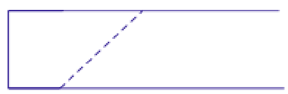
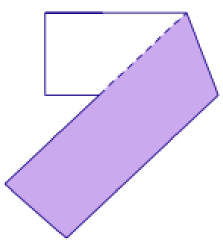
Iterative Folding[1]

1 Fold triangles

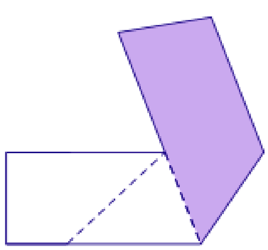
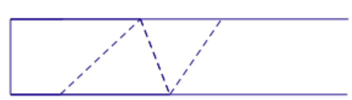
1. Take a long, thin strip of paper and begin by folding up to form any angle you like, then unfolding it to see the angle formed by the crease line and the bottom edge of the paper.

2. Now fold down so that the top edge of the paper falls along the crease line you just made, and unfold to see the new crease line.




Next fold up along the newest crease line, then unfold.

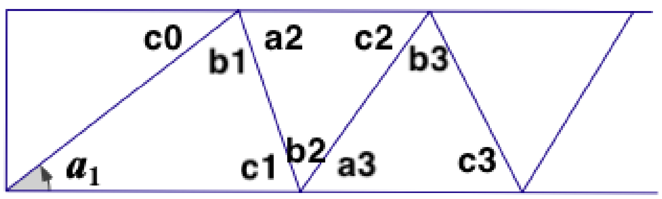
We will call this folding pattern U^1D^1. Notice that the crease line made by an "up" fold has positive slope, whereas one made by a "down" fold has a negative slope. Continue folding in this way, alternating folding up and down along the crease lines. What do you notice about the triangles that are formed by this procedure? Make a conjecture about these triangles.

[1]From Altha Rodin, Math Teachers Circle of Austin

2 Prove the conjecture by keeping track of angles

It appears that the triangles formed by the folding procedure are equilateral. Therefore, we would like to prove that the measure of the angles approaches 60° as the number of folds increases.

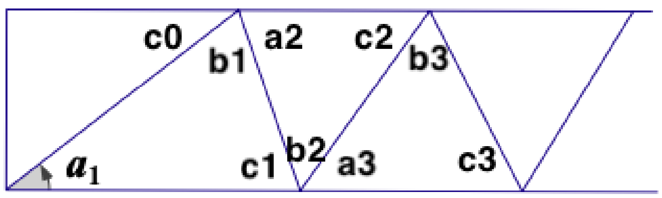
One way to prove the conjecture is to label the angles formed at each stage of the process, using as few different variables as possible. For example, if the angle created by the first fold is a_1, what can we say about angles formed by subsequent folds? What relationship will they have to a_1?



3. In the diagram above, what is the relationship between a_1 and c_0? Write c_0 in terms of a_1.
4. What is the relationship between b_1 and a_2? Write a_2 in terms of b_1.
5. What is the relationship between c_0, b_1, and a_2? Use this relationship to write b_1 and a_2 in terms of a_1.
6. How does c_1 compare to a_2? Use this relationship to write c_1 in terms of a_1.
7. How do b_2 and a_3 compare to c_1? Use this relationship to write b_2 and a_3 in terms of c_1.
8. Continue writing angles in terms of a_1 until you notice a pattern. Hint: instead of writing angles as number, write them out as sums and differences of things like $\frac{180°}{2}$, $\frac{180°}{4}$, $\frac{180°}{8}$, etc.
9. In the limit, what happens? If you don't know how to add up an infinite sum, ask for help!

3 Prove the conjecture by thinking about error

Another way to prove that the measure of the angles approaches 60° is to think about the error, or how far off the angles are from 60°.

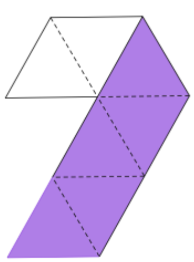


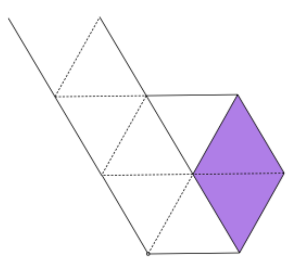
Suppose the angle a_1 is not exactly 60°, but is off by some amount, say ϵ. That is, a_1 is either ϵ more than 60° or ϵ less.

10. In terms of ϵ, how far off from 60° is c_0?
11. How far off from 60° are b_1 and a_2?
12. How far off from 60° is c_1?
13. How far off from 60° are b_2 and a_3?
14. How far off from 60° is c_2?
15. How far off from 60° are b_3 and a_4?
16. In the limit, how far off are the angles from 60°?

4 Folding a hexagon

Folding a hexagon: Now that we know that the measure of the angles we fold is approaching 60°, we can use the strip to fold a hexagon. Throw away the first few triangles on your strip of paper, then fold the paper along crease lines to form a hexagon, as illustrated below.

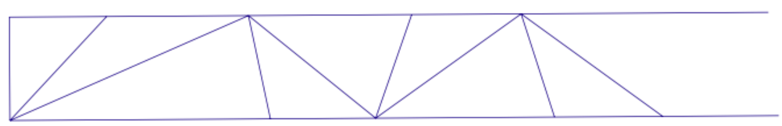




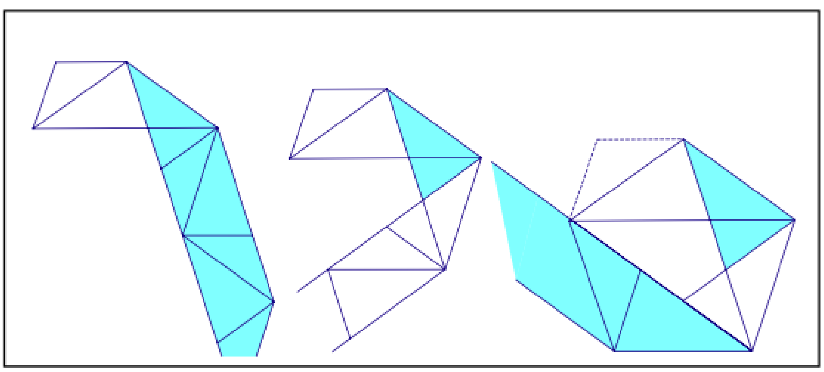
5 Generalize the idea

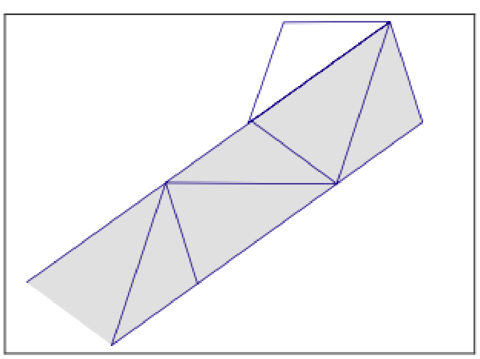
We may consider other folding patterns similar to the U^1D^1. One possibility is to fold in the same direction twice before changing the direction of the fold. We will call this folding pattern U^2D^2 . To perform the folding pattern, you fold up twice, then down twice, then repeat.

17. Take another long thin piece of paper and repeat the pattern U^2D^2 until you see the triangles beginning to stabilize.



18. It is a bit harder to see what angles are formed now, so we can use the idea of folding polygons to identify the angles that appear after a few iterations. As before, throw away the first few triangles on the strip. Try folding a regular polygon by folding along congruent crease lines. There will be two choices: you can fold only along short crease lines or you can fold only along long crease lines. How many sides do you get?





19. What does this tell you about the angles created by the folding pattern?

20. Can you prove your conjecture?

6 Generalize the idea further

Generalize the idea further: Can you find the angles formed by the U^3D^3 folding pattern? Can you prove your conjecture? Is it easier to fold polygons from the strip, or analyze the sequence of angles? Now consider the folding pattern U^2D^2 for an arbitrary positive integer n. What pattern do you notice relating the number of times you fold up (or down) and the measure of the angle you produce? What polygons can be formed from a U^nD^n strip, where n is any positive integer?

7 Generalize even more

Generalize even more: Try folding U^2D^1. Can you fold a regular polygon from this strip? Can you determine what polygon(s) can be folded from this strip by analyzing the sequence of angles produced? Note that we can fold in any sequence of "ups" and "downs". For example $U^2D^1U^3D^2$ means that we fold up twice, then down once, then up three times, then down twice. The sequence then repeats. Will this sequence "converge", (i.e., will the angles formed by the fold lines and the edges of the strip approach a set of fixed angles)?

8 More questions

Questions to consider:

21. Is there a folding sequence that will result in a strip that can be folded into a regular 7- gon?
22. Does every periodic folding sequence converge?
23. Can every regular polygon be folded using an iterative procedure to produce fold lines of the appropriate angle?

9 Acknowledgement

The material for this activity is taken from the book by Peter Hilton and Jean Pedersen, A Mathematical Tapestry: Demonstrating the Beautiful Unity of Mathematics, Cambridge University Press, 2010