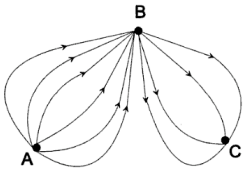
Combinatorics[1]

Product Rule and Sum Rule

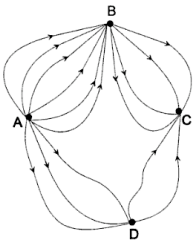
1. There are eight different flavors of ice cream and three different types of cones at the Yogurt Pump counter. You'd like to buy a different type of ice cream cone every day. How many days can you do this before you have to repeat?

   There are also 4 types of toppings. Now how many ways are there to buy a cone, ice cream, plus topping before you have to repeat?

2. There are three towns A, B, C, on Frog Island. There are 6 roads from A to B and 4 roads from B to C. How many ways can you drive from A to C (without going back through B!)



   A new town D is built and 5 new roads as shown. Now how many ways are there to drive from A to C?



Product Rule vs. Sum Rule:

(a) If we are counting the number of outcomes of a multi-stage process (like choosing from a sequence of menus), then the number of outcomes is the product of the number of choices for each stage.

(b) Whenever we separate the outcomes into several disjoint cases, each requiring separate counting methods, we add the number of outcomes in each case to get the total number of outcomes.

(c) Sometimes its easiest to count the number of outcomes we don't want and subtract that number from the total number of outcomes.

[1]Many of these problems are from *Mathematical Circles (Russian Experience). Others are from Paul Zeitz.*

3. A garden store has 4 types of cucumber plants, 7 types of tomato plants, and 3 types of watermelon. How many ways can you pick 2 different types of plants to put in your garden, if you don't want them both to be cucumbers, or both tomatoes, or both watermelons?

4. We call a natural number ultra-odd if all of its digits are odd. How many four-digit ultra-odd numbers are there? How many four-digit ultra-even numbers are there (i.e. all digits even)?

5. The Braille writing system uses 6 dots to represent the 26 letters of the English alphabet. Each dot can be either raised or flat. Are 6 dots also enough to represent the Japanese Katakana syllabic writing system, with 46 symbols?

6. There are 5 books on a shelf. How many ways are there to arrange three of them in a stack?

7. Seven children need to line up to wash their hands. How many different orders can they line up in? What if Hugo and Jed have to be next to each other. What if Hugo and Jed can't be next to each other?

8. There are 11 pies and 17 cakes for sale at the bakery. How many ways are there to choose one pie and one cake to serve for Thanksgiving? How many ways are there to choose 2 items of the same type (2 pies or 2 cakes)?

9. A code alphabet has only three letters: A, B, and C. A message in this code is an arbitrary sequence of no more than four letters, such as "CBB", or "ABAC". How many messages are possible?

10. Joan has 6 dogs. In how many ways can she choose one or more of them to accompany her on a hike?

11. You roll a 6-sided die 3 times. Among all possible outcomes, how many have at least two occurrences of the number 6?

Division rule and subtraction rule

12. How many ways are there to seat 6 people at a circular table, if the seating arrangement is considered the same if everyone has the same left neighbor and the same right neighbor?

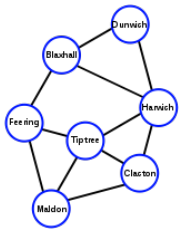
13. If a class contains 16 math majors and 12 comp sci majors, and no other students, and 5 of the students are double majoring in math and comp sci, how many students are in the class?

Division Rule and Subtraction Rule:

(a) When each item is counted with the same number of duplicates, divide by "the overcount", i.e. the number of duplicates.

(b) When only a few items are counted twice, subtract the number of duplicates.

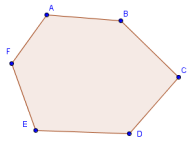
14. There are 20 towns on Planet X and every pair of them is connected by a road. How many roads are there? (There are no other roads besides the ones that connect the towns.)



15. How many ways are there to make a circular bracelet with 6 different beads, if all beads have to be used, but can be placed in any order?

16. How many 4 digit passcodes either start with the digit 0 or end with the two digits 11?

17. How many diagonals are there in a convex n-gon (a polygon with n-sides)? A diagonal is a line segment that connects two vertices that are not adjacent.



Permutations and Combinations

18. How many ways are there to pick a president, vice president, and treasurer for a club, if the club has 20 members.

19. How many ways are there to pick a team of 3 people from a club of 20 people?

For integers n and r with $0 \leq r \leq n$,

Permutations:

Permutations: The number of ways to arrange n different objects in a row is

$$n! = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1$$

The number of ways to arrange r objects in a row, when there are n distinct objects to choose from, is

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

Combinations:

The number of ways to choose r objects out of a collection of n objects is

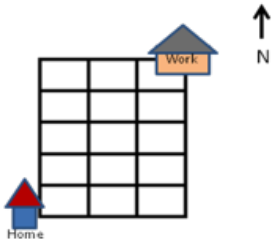
$$C(n,r) = \binom{n}{r} = \frac{n \cdot (n-1) \cdot (n-2) \ldots (n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$$

20. In how many ways can you choose team of 6 volunteers to staff a math festival from an applicant pool of 11 undergraduates and 17 grad students? What if the team has to consist of 3 undergrads and 3 grad students?

21. How many ways can you make a pizza with 3 different toppings if there are 8 toppings to choose from? What if the 3 toppings don't all have to be different? For example, you could have mushroom and double pepperoni as one option.

22. If you flip a coin 10 times, how many of the outcomes have exactly 4 heads and 6 tails?

23. How many different routes are there from home to work, only traveling north and east on streets?



24. A school has 100 students enrolled in fifth grade and 5 fifth grade teachers.

    (a) In how many ways can student assignments be made if each classroom holds 20 students?

    (b) Nora and Sophia are hoping to be in the same class in fifth grade. How many ways can student assignments be made in which this happens?

25. There are 5 time slots in a summer camp schedule, and one activity must be put into each slot to form a schedule. Each schedule must contain 3 sports activities, out of 8 to choose from, and 2 arts and crafts activities, out of 5 to choose from. How many schedules are possible?

26. How many ways can you choose a team from 11 people, where each team must have at least one person and the team must have a designated captain?

27. How many ways are there to arrange the letters in the word HOGWARTS? The letters in the word VOLDEMORT? The letters in the word ALOHOMORA? The letters in the words AVADAKEDAVRA?

28. How many ways are there to distribute 10 doggie biscuits among 7 dogs? The biscuits are indistinguishable, but the dogs are distinguishable.

Stars and Bars Rule
The number of ways to distribute n indistinguishable objects into k distinguishable boxes is given by
Anagram Rule
The number of ways to arrange n objects, when there are r_i indistinguishable objects of type 1, r_2 indistin objects of Type 2, ... and r_k indistinguishable objects of Type k, is given by:

29. You have 2 Reece's pieces, 4 mini bags of M&M's, and 1 sour candy left from Halloween. You are going to eat one candy per day until the candy is gone. How many different ways can you do this?

30. There are 10 questions on a math final exam. How many ways are there to assign point values to the problems if the sum of the scores is 100 and each question is worth at least 5 points?

31. How many ways can n books be placed on k distinguishable shelves

    (a) if the books are indistinguishable copies of the same title?

    (b) if no two books are the same, and the positions of the books on the shelves matters?

32. How many ways are there to represent the number 12 as a sum of

    (a) 5 non-negative integers?

    (b) 5 positive integers?

33. How many ways are there to rearrange the letters in the word "FLAMINGO" so that the vowels will be in alphabetical order and so will the consonants? For example, FAGILMON (A - I - O, F - G - L - M - N).

Challenge

34. How many ways are there to travel in xyzw space from the origin $(0, 0, 0, 0)$ to the point $(4, 3, 5, 4)$ by taking steps of one unit in the positive x, positive y, positive z, or positive w direction?

35. There are three rooms in a dormitory: one single, one double, and one for four students. How many ways are there to house seven students in these rooms?

36. You flip a coin 10 times. Of all the possible outcomes, how many have exactly 5 heads in a row? For example, we would not count HHHHHHHTTT (too many consecutive heads), but we would count TTTHHHHHTT and HHTTHHHHHT.

37. 10 boys and 9 girls sit in a row of 19 seats. How many ways can this be done if

    (a) All the boys sit next to each other and all the girls sit next to each other.

    (b) The children sit so that each person has only neighbors of the opposite sex.

38. In a traditional village, there are 10 young men and 10 young women. The village matchmaker arranges all the marriages.

    (a) In how many ways can they pair off the 20 young people, assuming a marriage has to be between a man and a woman?

    (b) In how many ways can they pair off the 20 young people if male-male couples, male-female couples, and female-female couples are all allowed?

39. How many ways are there to choose 6 cards from a complete deck of 52 cards in such a way that all four suits will be present?

40. Consider the following poker hands:

    (a) Royal flush: A, K, Q, J, 10, all the same suit.

    (b) Straight flush: Five cards in a sequence, all in the same suit.

    (c) Four of a kind: All four cards of the same rank.

    (d) Full house: Three of a kind with a pair.

    (e) Flush: Any five cards of the same suit, but not in a sequence.

    (f) Straight: Five cards in a sequence, but not of the same suit.

    (g) Three of a kind: Three cards of the same rank.

    (h) Two pair: Two different pairs.

    (i) Pair: Two cards of the same rank.

    Find the probability of each kind of hand.