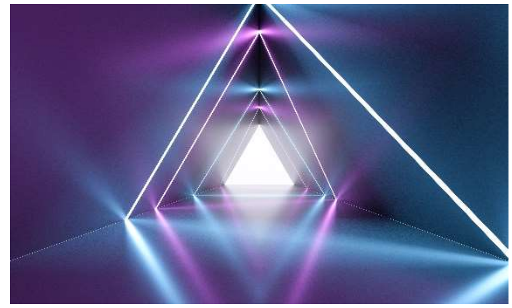
Chapel Hill Math Circle
Session 11 – March 2, 2024:
Triangles, Triangular Numbers, & Handshakes
Beginners' Group (grades 1-3), 10:30-11:30a
Mr. Barman – dilip@trianglemathinstitute.com



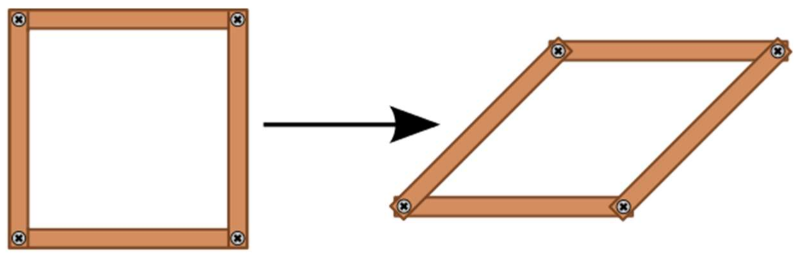
Supplies needed per person: 7 wooden popsicle sticks, masking tape, 50-60 coins or other tokens that are all the same size

Happy March! Let's look at triangles, numbers in the shape of triangles, and handshakes today. Did you know that triangles are among the strongest shapes? Try this activity.

Build a rectangle and then a triangle – what do you notice when you try to deform them?

Use four popsicle sticks at home if you have them; we don't so today we'll use pencils (you can also use toothpicks). Be careful about the sharpened ends (of pencils and both ends if using toothpicks). Use masking tape at four joints to make a rectangle. Wrap the masking tape tightly so that the rectangle holds.

Now put one side of the rectangle on your table and press down on the opposite side. It will easily collapse or "shear" and fall apart. Remake the rectangle and push left to right or right to left as shown here and it will also deform.



But what happens when you do something like this with a triangle? What do you predict? Try it! Make a prediction and write it in the box below and then note what actually happens. Then note in the box why you think this happens.

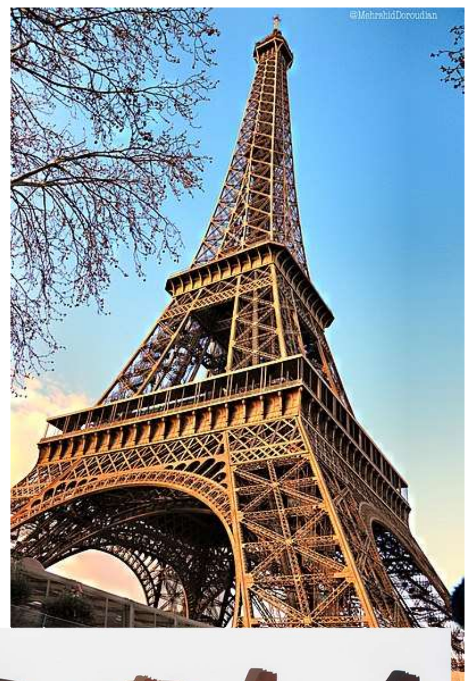
Prediction of what happens with a triangle when you press down or to the side?

What actually happens?

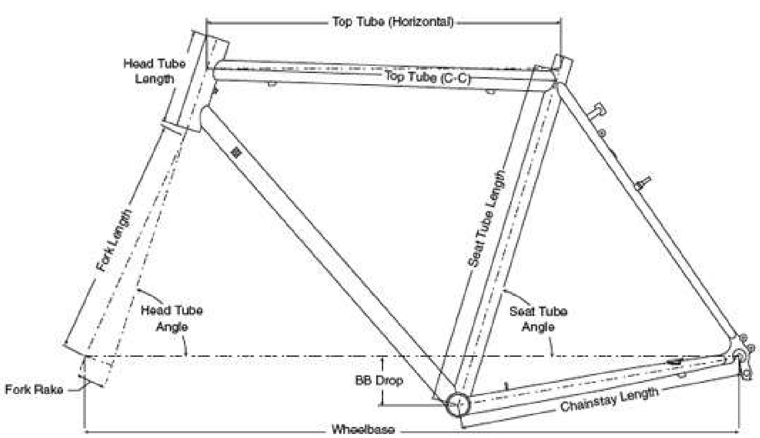
Why do you think this happens?

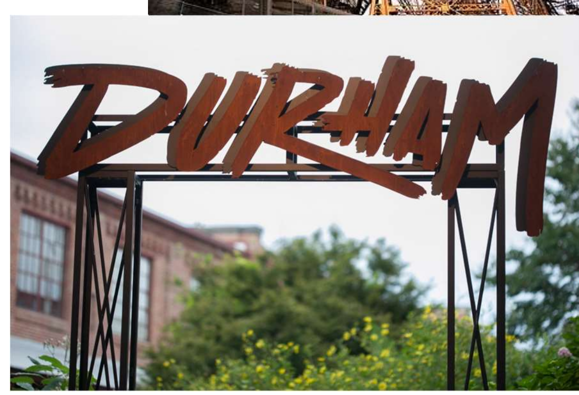
Triangles gives “rigidity” or stiffness to a shape. You can bear down on a triangle and, as long as the sides are reasonably strong, it won’t break apart unless you break the joints. Pushing down or to the side causes the “force” to be spread out.

If you are a woodworker, construction worker, carpenter, or other craftsperson or builder, you will want to plan the angles that you use very carefully. Engineers like to use triangles in structures to give them more strength. Look at the trusses that go under the Golden Gate Bridge – triangles to give the bridge more strength. (The pictures are from the Golden Gate Highway & Transportation District[1].) Do you see triangles in the San Francisco-Oakland Bay Bridge picture on the next page[2]?



Bicycle frames are strengthened with diamonds (two triangles!) and other angles[3].
Where can you find triangles to add strength to something? See if you can find supporting triangles in buildings (see the Eiffel Tower image[4]), scaffolding (I took that picture of the Durham sign at the American Tobacco Campus on October 5, 2019), musical instruments, toys, and more. Do the triangles add strength to the object?





[1] goldengate.org/exhibits/how-the-bridge-spans-the-golden-gate, accessed Nov. 28, 2023
[2] from mtc.ca.gov/operations/programs-projects/bridges/san-francisco-oakland-bay-bridge, accessed Nov. 28, 2023
[3] This diagram is from cyclingabout.com/understanding-bicycle-frame-geometry, accessed Nov. 28, 2023
[4] commons.wikimedia.org/wiki/File:Eiffel_tower_2.jpg

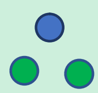
Can you find triangles in art? In food? Here is a picture of a meal that I served in February 2011; I am always mindful of shapes and designs in my plating of food.

Triangular Numbers

Let’s explore Triangular Numbers. You will need 50-60 coins, tokens, cereal pieces, or something else. It’s best if the pieces are fairly similar to each other.

Your goal is to make triangles with these objects – I will use dots. You want the objects (I’ll say “dots” but you may be using something else) to be the same distance apart from each other. Isn’t the arrangement to the right the first such triangle that you can make? Looking at what we made, we call the first Triangular Number 1+2 = 3. Let’s give it a simple, boring, but easy name – T with a little 1 – like this: T_1.

Use the tokens or just your head and draw and count the next few Triangular Numbers. In the box below draw the next few Triangular Numbers; if you have colored pencils, color each row a different color. Leave T_0 blank for now. Once you get the pattern you can stop drawing dots but still show your sums.

T_0:	**T_1:**  1+2 = 3	**T_2:**	**T_3:**
T_4:	**T_5:**	**T_6:**	**T_7:**
T_8:	**T_9:**	**T_{10}:**	**T_{11}:**

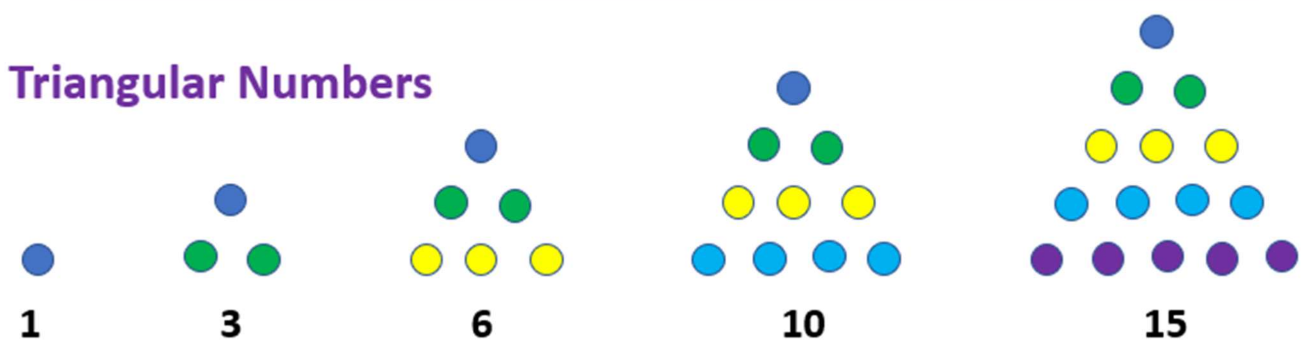
Math people like us enjoy finding patterns. Can you predict what T_0 is based on T_1, T_2, T_3, … ?

How do we calculate bigger Triangular Numbers?

There was a famous mathematician named Carl Friedrich Gauss[5] (history remembers him as "Gauss", which is what I'll call him). He was from Germany and lived from 1777-1865. He created a whole new type of math called graph theory (I hope to share a glimpse of it in a future math circle) – and he did so much more including with astronomy.

As you can imagine he was very smart. One day his math teacher asked him, back when he was around your age, to find the 100th Triangular Number. She thought that this would occupy him for a while. You can imagine her surprise when he quickly responded 5050. How did he do it?

Clearly Gauss didn't go in order and add 1+2 and then add on 3 and then add on 4 and so forth. Do you have any ideas? Experiment with a number like T_{10} in the box below; is there an easier way to find its value other than the obvious way of adding left to right?



[5] I copied his picture from britannica.com/biography/Carl-Friedrich-Gauss, accessed on March 1, 2024.

How about what I call the Rainbow Method ?

Let me just use the method and then we can discuss it. What do you think of this way of finding T_{10}?

$$T_{10} = 1+2+3+4+5+6+7+8+9+10$$

Once we do the (not so) hard work of adding the smallest and largest numbers, 1+10, don't all of the other pairs of next largest-next smallest have to be the same? I don't really have to think but right away know that 2+9 is also 1+10, as is 3+8, as is …. So we have 5 11s; 11+11+11+11+11 = 55, which is the value of T_{10}. That was much more fun, less error-prone, and more interesting than the straightforward way of finding T_{10}, wasn't it?

So T_{100} is just 50 pairs of 101 (1+100 = 2+99 = 3+98 = … = 50+51 = 101). 50 100s is 5000 and 50 1s is 50, so we have T_{100} = 1+2+3+4+ … + 100 = 5050, just as Gauss said!

But what do we do if we have an odd number? Try recalculating a few small Triangular Numbers, T_3 and T_5 to see if you can come up with a strategy, and then try finding T_{101}. If you are feeling up to the task, you can try T_{201}, and T_{1001}.

Did you get these answers?   3 5 101 201 1001

$$T_3 = 1 + 2 + 3$$

$$T_5 = 1 + 2 + 3 + 4 + 5$$

$$T_{101} = 1 + 2 + 3 + \dots + 50 + 51 + 52 + \dots + 101$$

Now the pattern seems clear, no?

- T_3 = 4+2 = 6
- T_5 = (1+5)+(2+4)+3 = 6+6+3 = 15
- T_{101} = 102+102+102+…+102 {50 times} + 51 = 5000+100+51 = 5151
- Okay, so for T_{201} we will have 100 pairs of 202s plus 101 = 20,200+101 = 20,301
- For the 1001st Triangular Number we have 500 pairs of 1002s plus 501
  - 500 pairs of 1002s … well, 100 pairs is 100,200, so 500 pairs is 501,000
  - So our total is 501,000+501 = 501,501

Wow! It's okay if you didn't get the last few. The important thing is to get the pattern.

It can be a little tricky keeping track of what pairs we are adding. Here is another way to do this. Can't we rewrite our sums backwards? If we then add, we are automatically adding the smallest and largest then next smallest and next largest, etc. But we have twice of what we're seeking so we take half. Here's a small then a big example.

$$T_5 = 1 + 2 + 3 + 4 + 5$$
$$T_5 = 5 + 4 + 3 + 2 + 1$$
$$T_5 + T_5 = 6 + 6 + 6 + 6 + 6 = 30; \quad T_5 = 15$$

$$T_{101} = 1 + 2 + 3 + \dots + 101$$
$$T_{101} = 101 + 100 + 99 + \dots + 1$$
$$T_{101} + T_{101} = 101 \text{ } 102\text{s}; \quad T_{101} = 101 \text{ } 51\text{s} = 100 \text{ } 51\text{s} + 51 = 5151$$

Square Numbers?

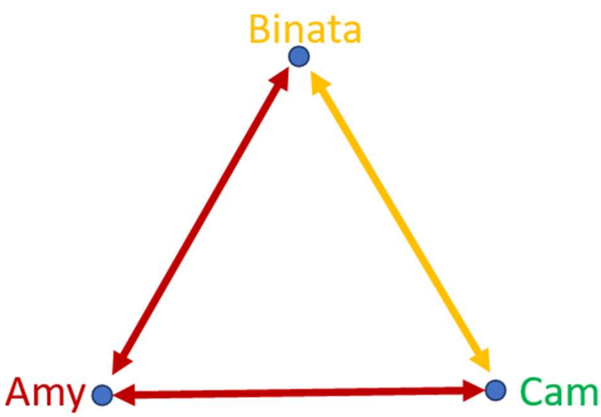
That was fun! We'll stop with Triangular Numbers for now but you might want to think about what Square Numbers are.

The Handshake Problem

Let's say that we have some people who want to shake each other's hands. The rules are:

- You can't shake your own hand
- Each pair of different people must shake hands once

For example, if Amy, Binata, and Cam meet, the handshakes are Amy-Binata, Amy-Cam, and Binata-Cam. There are 3 handshakes and everybody has shaken everybody else's hand once.



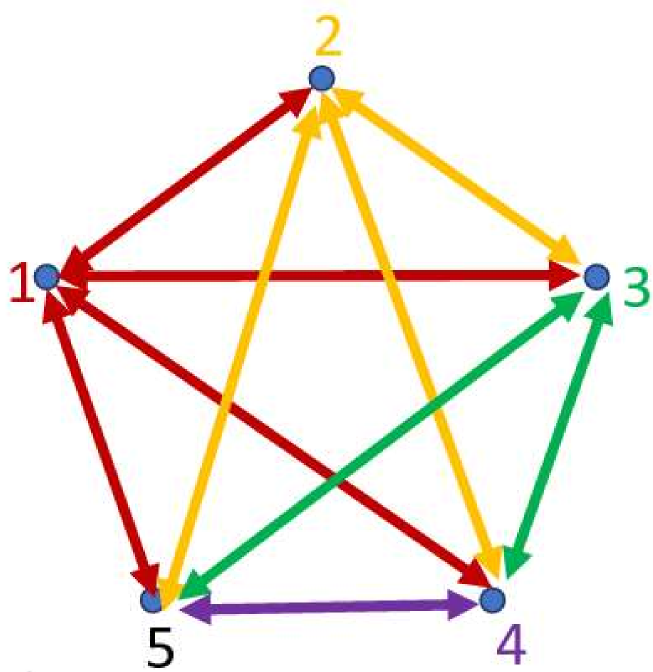
We can show this via a picture. One picture shows this while the other shows all the handshakes among 5 friends. For 5 friends, there are 4+3+2+1 = 10 handshakes, right?

If there are 2 people there's 1 handshake.

If there are 3 people there are 3 handshakes.

If there are 4 people (A shakes hands of B, C, and D; B shakes hands of C and D; and C shakes hands of D) there are 6 handshakes.

Does 1, 3, 6, 10, … look familiar? Yes - they are Triangular Numbers!

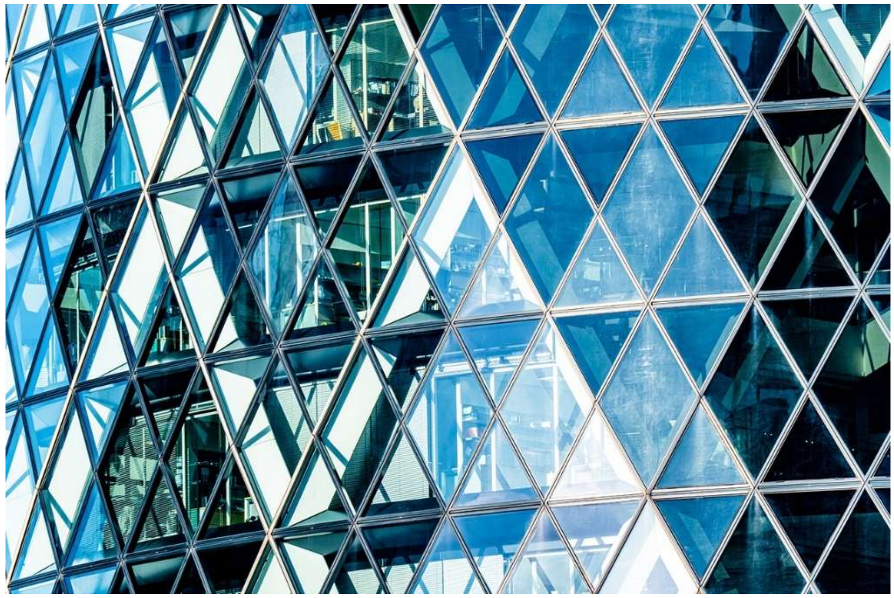


Activities for you to consider

- How many handshakes are there if there are 100 people in a room?
- Try the Handshake Problem at home with your family. Try shaking hands and seeing if the number of handshakes is what you expect it to be.
- Look for triangles – everywhere!
- What are Square Numbers? Is there a way to relate Square and Triangular Numbers?

Have Fun!
Mr. Barman

Notes for Parents



Triangles are all around us. In fact our area of Chapel Hill-Durham-Raleigh is called “The Triangle” – a triangle of cities. Nearby is “The Triad” of Winston-Salem-Greensboro-High Point. Triangles are important in many of the world’s religions and philosophies. And they’re fundamental to mathematics.

It’s easy to find references to the strength of triangles. One quick article on medium.com[6], for example, has images that you can show to your child. Unless one of the vertices connecting the three line segments is broken, a triangle is unlikely to break or lose its basic shape.

You might enjoy reading a bit more about the true story of Gauss. One good article is “Gauss’s Day of Reckoning” by Brian Hayes in *American Scientist*[7].

I hope that your child will continue exploring these problems. Don’t worry if some of the later few problems were a bit arithmetically adanced; if your child understands the pattern then that’s all that’s important.

By the way, I’ve been toying with the idea of putting together a math cruise. I give talks on vegan cruises so why not create a math cruise? The idea is that we’d have a 5-7 day cruise mostly with fun, free time. I’d have perhaps a 2-hour math circle in the morning for younger children and a similar session in the afternoon for older children. We could have math games, a Julia Robinson math festival (check out jrmf.org; I’m on their Teacher Advisory Council), math films, and a lot of fun. Let me know if you are interested.

Till next time!

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[6] medium.com/@hristovh1980/the-best-shape-for-making-strong-structures-d558ff3b876a

[7] americanscientist.org/article/gausss-day-of-reckoning