An Introduction to Mass Point Geometry, Part 1 of 2

Abstract

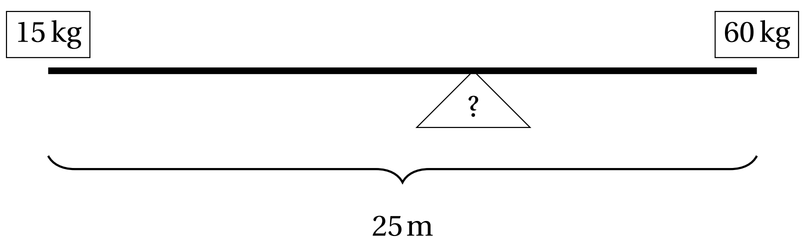
In this session, we shall introduce *mass point geometry*. Motivation comes from physics, generalizing the notion of the center of mass. Mass point geometry techniques can be powerful, offering much simpler ways to solve a number of problems from classical geometry, both in the plane and in higher dimensions. Our approach is modeled on that of [12] its subsequent adaptation in [10].

Background needed: We assume basic familiarity with geometry, including describing points in $\mathbb{R}^n$ as n-tuples of the form $(x_1, x_2, \ldots, x_n)$, and making algebraic computations associated with such a description of points.

0 Warmup

Exercises:

0.1 Consider a rigid bar, like a seesaw, of length 25 m and negligible mass. We place masses 15 kg and 60 kg at each respective end:



In order for the system, including these weights at those positions, to balance parallel to the ground, where should we place the fulcrum? (Don't worry about a formal proof for *why* your selection for the location of such a pivot point would work.)

1 Preliminaries: Definitions and Notation

We begin with some notation and definitions:

Definition 1.1. Let P be a point, and $m > 0$ a real number. Then a *mass point* is an expression denoted mP. Two mass points mP and nQ are defined to be equal if and only if $m = n$ and $P = Q$. We shall often simply write "P" as shorthand for the mass point $1P$, too, provided the context is clear that this represents a mass point and not simply a point.

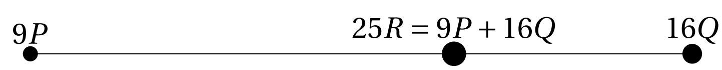
Definition 1.2. Let mP and nQ be two mass points. The *mass point sum* of mP and nQ, denoted $mP + nQ$, is defined as follows:

- If $P = Q$, then $mP + nQ := (m+n)P$.
- If $P \neq Q$, then $mP + nQ$ is defined to be mass point of mass $m + n$ at the unique point R on the line segment $\overline{PQ}$ that lies $n/(m+n)$ of the distance PQ from P to Q. That is,

$$\frac{PR}{RQ} = \frac{n}{m}. \tag{1.1}$$

  If $mP + nQ = (m+n)R$, we say that $\overline{PQ}$ is *balanced at* R relative to the mass points mP and nQ.
- If mP is a mass point and $a > 0$, then we define $a(mP) := (am)P$.

Intuitively, the point R is the center of mass for the set $\{mP, nQ\}$ of mass points. If you prefer, imagine, as in Exercise 0.1, a balance beam with the given masses at the respective points. Its balancing point is at R, and the mass at point R is $m + n$, the sum of the masses of the other given points. See the example below:



The mass point sum $mP + nQ$ then represents a kind of summary of the entire system, encoding both *total mass* and the location of the *center of mass*. Namely, the mass point $mP + nQ$ sum has mass $m + n$, the sum of the two given masses. Further, its location is at the center of mass of the two given masses at their respective locations.

Note in particular that the mass point sum is *closer* to the point with *larger* mass. When $m = n$, so that the masses are equal, then the mass point $mP + nQ = mP + mQ$ lies at the midpoint of the segment $\overline{PQ}$.

Proposition 1.3. *Let ℓO, mP, and nQ be any mass points, and assume $a > 0$ Then*

- *Mass point addition is commutative: $mP + nQ = nQ + mP$.*
- *Mass point addition is associative: $\ell O + (mP + nQ) = (\ell O + mP) + nQ$*
- *Mass point addition is distributive: $a(mP + nQ) = amP + anQ$.*

All these properties can (and should!) be justified, of course. (See, for example, Exercises #5.4 below.) For now, though, let's accept these provisionally and use the properties of mass points and their sums to solve certain exercises.

2 Working with Mass Point Sums

Exercises: Throughout, your solutions need not be drawn precisely to scale. So long as you can explain what's going on correctly, and generally explain where the point should appear, that will suffice.

2.1 Draw the following mass point sum $mP + nQ$:

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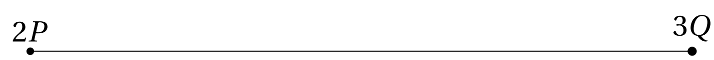
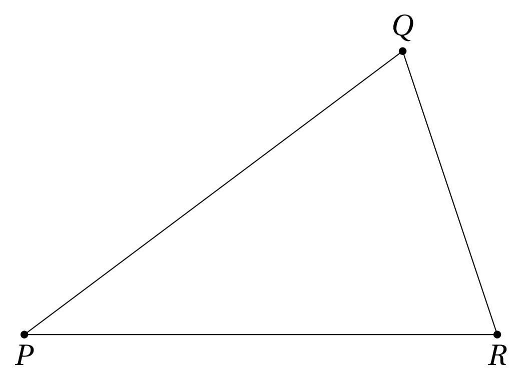


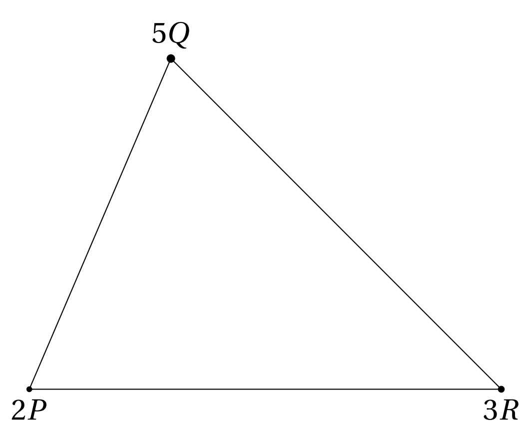
2.3 Draw the following mass point sum $mP + nQ$:

2.4 Draw the following mass point sum $mP + nQ$:

2.5 Draw the mass point sum $1P + 1Q + 1R$ indicated below:



2.6 Draw the mass point sum $2P + 5Q + 3R$ indicated below:



3 Basic Strategies with Mass Points

Strategy 3.1. *The following are strategies for using mass points:*

(a) *Assume that $\overline{PQ}$ is a line segment containing the point R. Then if we know $PR/RQ = n/m$, assign masses m and n to P and Q respectively so that $mP + nQ = (m+n)R$. Further, we can scale this by a positive constant k to have $k(m+n)R = kmP + kmQ$, as well.*

Recall that the numerator in this ratio is assigned to Q and the denominator is assigned to P, so that the larger *mass is assigned to which of P or Q is* closer *to R.*

(b) *Given mass points mP and nQ, use mass point operations to determine the location of R on $\overline{PQ}$ such that $mP + nQ = (m+n)R$.*

(c) *Let P, Q, and R be points. If for some masses m and n we have that $mP+nQ = (m+n)R$, then R must lie on segment $\overline{PQ}$.*

That is, if we can show that a particular point R is the location of a mass point sum of the form $mP+nQ$, then we can conclude that R lies on $\overline{PQ}$. This will be especially useful in showing two lines intersect in a particular *point, or that three lines are collinear.*

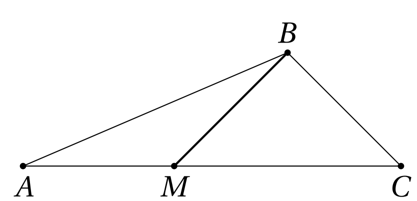
(d) *We can "split" a single mass point, representing it as the sum of two mass points corresponding to the same underlying point.*

For example, a given mass point $(m+n)P$ can be rewritten in the equivalent form $mP + nP$. (This will be easier to appreciate with examples; see Exercises #5.1 and #12.2.

In particular, Strategies 3.1(b)–3.1(c) are often useful to show that that *three* line segments intersect in a common point. Strategy 3.1(d) is invaluable when our desired assignment of mass points would otherwise be inconsistent with the given hypotheses.

4 Cevians and Mass Points

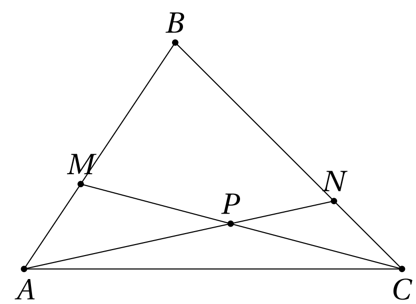
Definition. Let $\triangle ABC$ be any triangle. A *cevian*[1] is a line segment connecting one of the vertices of the triangle with any point (excluding the endpoints) of the opposite side. For example, $\overline{BM}$ below is a cevian of $\triangle ABC$.



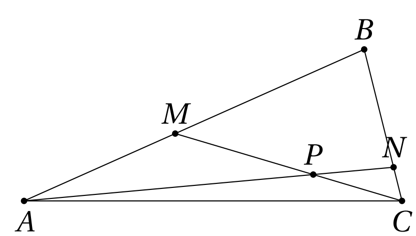
[1] This term named for Italian mathematician Giovanni Ceva. Following Italian pronunciation rules, it would therefore be pronouced "CHAY-vee-uhn".

4.1 Consider a triangle $\triangle ABC$. A *median* of a triangle is a line segment whose endpoints are one of the triangle's vertices and the midpoint of the opposite side. Prove that the three medians of any triangle intersect in a common point, called the *centroid* of a triangle. Further, prove that the medians divide each other in the ratio 2-to-1, where the point of intersection lies farther from each vertex than from the opposite site.

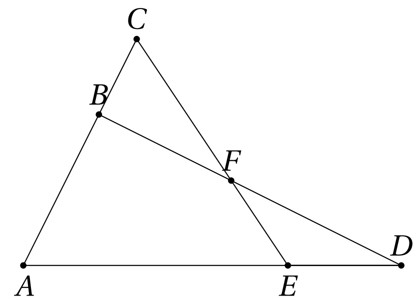
4.2 Consider $\triangle ABC$, with cevians $\overline{AN}$ and $\overline{CM}$ that intersect in a common point P, as below. If $AM/MB = 3/5$ and $BN/NC = 7/3$. Compute the ratios AP/PN and CP/PM.



4.3 Consider $\triangle ABC$, with cevians $\overline{AN}$ and $\overline{CM}$ that intersect in a common point P, as below. If $AM/MB = 4/5$ and $BN/NC = 7/2$. Compute the ratios AP/PN and CP/PM.

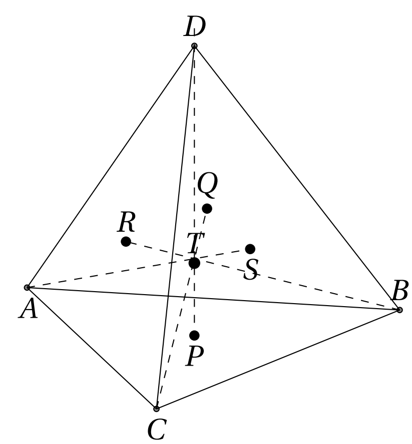


4.4 Consider the diagram below. If $AB/BC = 2$ and $AE/ED = 7/3$, then compute BF/FD and CF/FE.



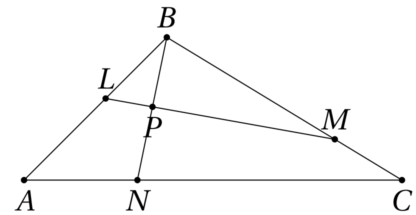
4.5 Consider a tetrahedron $ABCD$ in space. Let P, Q, R, and S be, respectively, the centroids of $\triangle ABC$, $\triangle ABD$, $\triangle ACD$, and $\triangle BCD$. (See Exercise #4.1 for the definition of the centroid of a triangle.) Prove that the line segments $\overline{AS}$, $\overline{BR}$, $\overline{CQ}$, and $\overline{DP}$ all intersect in a common point T. What are the ratios AT/TS, BT/TR, CT/TQ, and DT/TP?

Can you generalize this result to polyhedra in dimension 4 or higher?



5 Additional Exercises

5.1 Consider $\triangle ABC$, with cevian $\overline{BN}$ and transversal $\overline{LM}$ that intersect in a common point P, as below. If $AL/LB = 4/3$, $BM/MC = 5/2$, and $CN/NA = 7/3$, then compute the ratios LP/PM and BP/PN.



5.2 Given mass points mP and nQ, how might you define the mass point difference $mP - nQ$? Under what conditions would $mP - nQ$ exist?

5.3 Say that mP and nQ are mass points, where in terms of Cartesian coordinates, $P := (x_1, y_1)$ and $Q := (x_2, y_2)$. What are the Cartesian coordinates of the point R, where $mP + nQ = (m+n)R$? If we are in three-dimensional space and the points have coordinates given by $P := (x_1, y_1, z_1)$, $Q := (x_2, y_2, z_2)$?

Hint: If the general case seems too abstract, consider the following special case. Given distinct points $P := (x_1, y_1, z_1)$ and $Q := (x_2, y_2, z_2)$, what are the coordinates of the midpoint M of the segment $\overline{PQ}$? What about the point R on $\overline{PQ}$ that lies one-fourth of the way from P to Q on this segment?

5.4 Earlier, we asked you to use Proposition 1.3 above without yet justifying it. Now, prove each of its three claims.

An Introduction to Mass Point Geometry, Part 2 of 2

Abstract

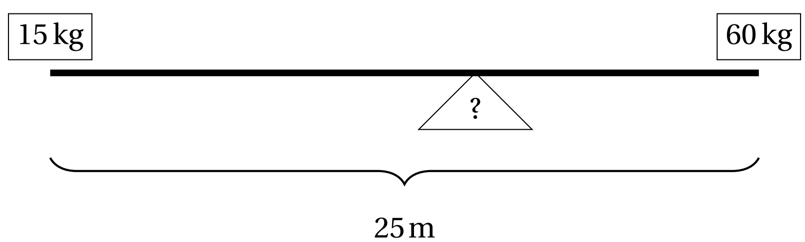
In this session, we shall continue our previous exploration of mass point geometry. Motivation comes from physics, generalizing the notion of the center of mass. Mass point geometry techniques can be powerful, offering much simpler ways to solve a number of problems from classical geometry, both in the plane and in higher dimensions. Our approach is modeled on that of [12] its subsequent adaptation in [10].

Background needed: As with Part 1, we assume basic familiarity with geometry, including describing points in $\mathbb{R}^n$ as n-tuples of the form $(x_1, x_2, \ldots, x_n)$, and making algebraic computations associated with such a description of points. Several of the exercises below also assume some familiarity with trigonometry.

6 Warmup, Part 2

Exercises:

6.1 Consider a rigid bar, like a seesaw, of length 25 m and negligible mass. We place masses 15 kg and 60 kg at each respective end:



In order for the system, including these weights at those positions, to balance parallel to the ground, where should we place the fulcrum? (Don't worry about a formal proof for *why* your selection for the location of such a pivot point would work.)

7 Review: Mass Point Geometry

Let's recall some of the basic definitions and results about mass point geometry which we introduced in our last session. Our approach is modeled on that of [12] its subsequent adaptation in [10].

Definition 7.1. Let P be a point, and $m > 0$ a real number. Then a *mass point* is an expression denoted mP. Two mass points mP and nQ are by definition equal if and only if $m = n$ and $P = Q$. We shall often simply write "P" as shorthand for the mass point $1P$, too, provided the context is clear that this represents a mass point and not simply a point.

Definition 7.2. Let mP and nQ be two mass points. The *mass point sum* of mP and nQ, denoted $mP + nQ$, is defined as follows:

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$$\frac{PR}{RQ} = \frac{n}{m}. \tag{7.1}$$

  If $mP + nQ = (m+n)R$, we say that $\overline{PQ}$ is *balanced at R* relative to the mass points mP and nQ.
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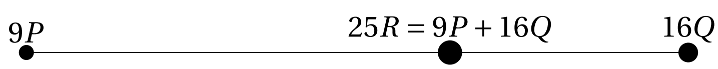


Note in particular that the mass point sum is *closer* to the point with *larger* mass. When $m = n$, the mass point $mP + nQ = mP + mQ$ lies at the midpoint of the segment $\overline{PQ}$.

Proposition 7.3. *Let ℓO, mP, and nQ be mass points, and assume $a > 0$ Then*

- *Mass point addition is commutative: $mP + nQ = nQ + mP$.*
- *Mass point addition is associative: $\ell O + (mP + nQ) = (\ell O + mP) + nQ$*
- *Mass point addition is distributive: $a(mP + nQ) = amP + anQ$.*

Proving both commutativity and distributivity is relatively straightforward. The associativity of mass point addition, however, is more complicated.

The following are strategies for using mass points:

Strategy 7.4. *The following are strategies for using mass points:*

7.4.1 *Assume that $\overline{PQ}$ is a line segment containing the point R. Then if we know $PR/RQ = n/m$, assign masses m and n to P and Q respectively so that $mP + nQ = (m+n)R$. Further, we can scale this by a positive constant k to have $k(m+n)R = kmP + kmQ$, as well.*

Recall that the numerator in this ratio is assigned to Q and the denominator is assigned to P, so that the larger *mass is assigned to which of P or Q is* closer *to R.*

7.4.2 *Given mass points mP and nQ, use mass point operations to determine the location of R on $\overline{PQ}$ such that $mP + nQ = (m+n)R$.*

7.4.3 *Let P, Q, and R be points. If for some masses m and n we have that $mP+nQ = (m+n)R$, then R must lie on segment $\overline{PQ}$.*

That is, if we can show that a particular point R is the location of a mass point sum of the form $mP+nQ$, then we can conclude that R lies on $\overline{PQ}$. This will be especially useful in showing two lines intersect in a particular *point, or that three lines are collinear.*

7.4.4 *We can "split" a single mass point, representing it as the sum of two mass points corresponding to the same underlying point.*

For example, a given mass point $(m+n)P$ can be rewritten in the equivalent form $mP + nP$. (This will be easier to appreciate with examples; see Exercises #5.1/#10.1 and #12.2.)

8 Worked Examples

To best illustrate how to work with mass points, we present the following examples with solutions.

Example 8.1. For $\triangle PQR$ as in Figure 8.1, compute the mass point sum $2P+5Q+3R$.
Solution, Method 1: Using the associativity of mass point sums from Proposition 7.3, in particular, we have $2P+5Q+3R = (2P+5Q)+3R$; that is, we first compute the mass point sum $2P+5Q$, then take the the mass point sum of that and $3R$.

For $2P+5Q$, this is a mass point located at the point S that is $5/(2+5) = 5/7$ of the way from P to Q along $\overline{PQ}$, and it has mass $2+5=7$. The overall mass point sum $2P+5Q+3R = (2P+5Q)+3R = 7S+3R$ lies at the point V that is $7/(7+3) = 7/10$ of the way from R to S along $\overline{RS}$. See Figure 8.2a for details. □
Solution, Method 2: This time, compute $2P+5Q+3R$ as the mass point sum $2P+(5Q+3R)$. For the mass point sum $5Q+3R$, we see this is located at the point T that is $5/(5+3) = 5/8$ of the way from R to Q along $\overline{RQ}$, and it has mass $5+3=8$. The overall mass point sum $2P+5Q+3R = 2P+(5Q+3R)$ is located at the point V that is $8/(2+8) = 4/5$ of the way from P to T along $\overline{PT}$. See Figure 8.2b for details. □
Solution, Method 3: Finally, we compute $2P+5Q+3R$ as the mass point sum $(2P+3R)+5Q$. Balance $\overline{PQ}$ at U, the point $3/(2+3) = 3/5$ of the way from P to Q along $\overline{PQ}$. Then $2P+3R =$

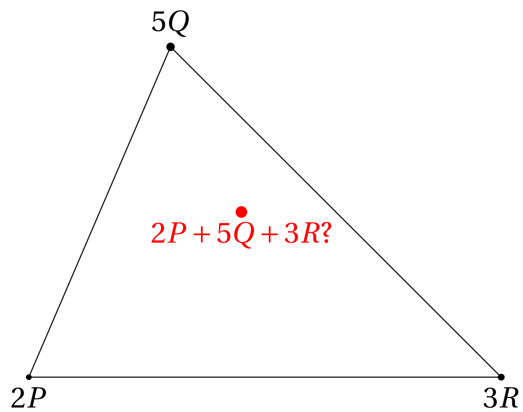
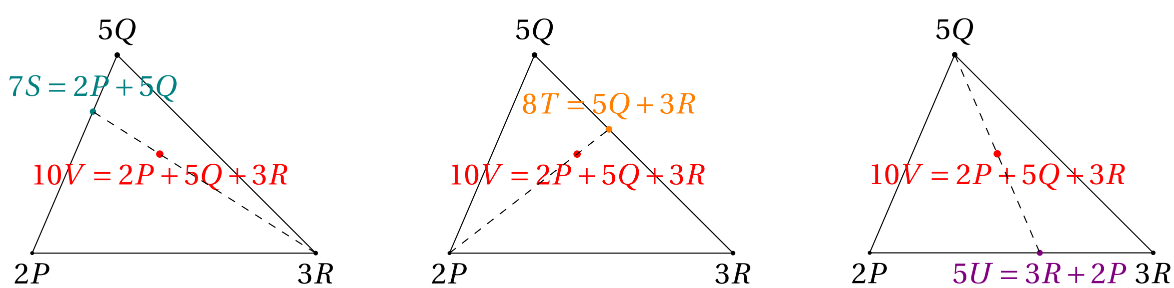


Figure 8.1: Computing $2P+5Q+3R$. Where is this mass point sum, and what mass is assigned there?



(a) Computing $2P+5Q+3R$ as $(2P+5Q)+3R$.

(b) Computing $2P+5Q+3R$ as $2P+(5Q+3R)$.

(c) Computing $2P+5Q+3R$ as $(2P+3R)+5Q$.

Figure 8.2: Computing $2P+5Q+3R$ in three different ways.

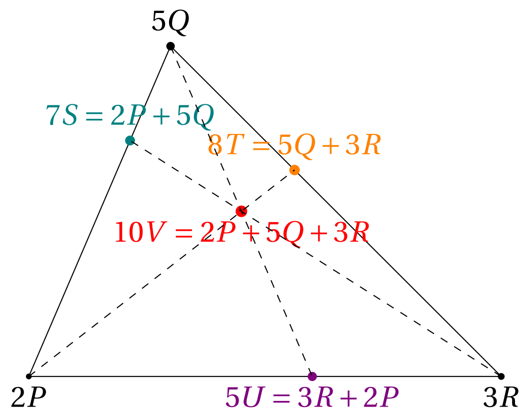
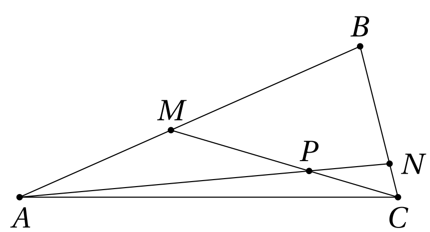


Figure 8.3: Computing $2P+5Q+3R$ in three different ways, overlaid.

$(2+3)U = 5U$, and $2P+5Q+3R = (2P+3R)+5Q$, balancing $\overline{QU}$ at V, where V is $5/(5+5) = 1/2$ of the way from U to Q along $\overline{QU}$; see Figure 8.2c for details. □

To illustrate that these three methods agree, so that the cevians in Figures 8.2a–8.2c intersect in the common point V, we overlay these as Figure 8.3:

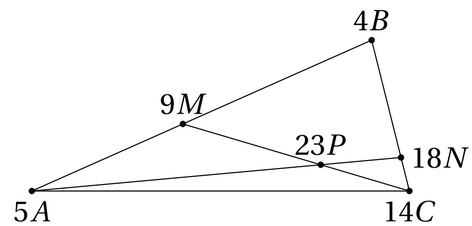
Example 8.2. Consider $\triangle ABC$, with *cevians*[2] $\overline{AN}$ and $\overline{CM}$ that intersect in a common point P, as below.[3] If $AM/MB = 4/5$ and $BN/NC = 7/2$. Compute the ratios AP/PN and CP/PM.



Solution: Considering $AM/MB = 4/5$, use Strategy 7.4.1, and assign masses to A and B to produce the mass points $5A$ and $4B$. We also have $BN/NC = 2/7$. Since we have already assigned mass 4 to B, for B and C to balance at N, assign mass $4 \cdot \frac{7}{2} = 14$ to C so that $4B + 14C = (4+14)N = 18N$, whence $BN/NC = 14/4 = 7/2$, as desired.

[2]A *cevian* of a triangle $\triangle ABC$ is a line segment, one of whose endpoints is a vertex of $\triangle ABC$, and where the other endpoint is any point of the opposite side, excluding the endpoints. This term named for Italian mathematician Giovanni Ceva. (Following Italian pronunciation rules, it would therefore be pronouced "CHAY-vee-uhn".)

[3]This was Exercise #4.3 from the previous session



Then considering the mass point sum $5A+4B+14C$, this lies at the point P, with $5A+4B+14C = 5A+(4B+14C) = 5A+18 = (5A+4B) = 9M+14C = 23N$. Therefore, since $\overline{AN}$ is balanced at P, $\underline{AP/PN = 18/5}$. Similarly, since $\overline{CM}$ is also balanced at P, $\underline{CP/PM = 9/14}$. □

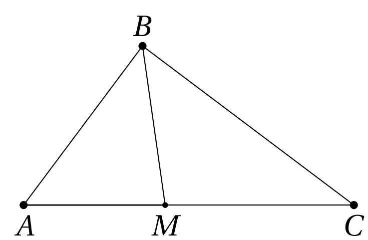
9 Geometric Background: The Law of Sines, Angle Bisectors, and Areas

Let us review some preliminary results which will be useful in combination with techniques from mass point geometry. These can be proven *without* requiring mass point geometry, but Exercise #9.3 (the Law of Sines) will require some basic familiarity with trigonometry.

Notation. If $\triangle ABC$ is a triangle, then unless otherwise indicated, we let A, B, and C denote angles $\angle BAC$, $\angle ABC$, and $\angle ACB$, respectively. Further, we let a, b, and c denote the lengths BC, AC, and AB, respectively.

9.1 Let $\triangle ABC$ be a triangle. If M lies on $\overline{AC}$, then $\overline{BM}$ bisects $\angle ABC$ if and only if

$$\frac{c}{a} = \frac{AM}{MC}. \tag{9.1}$$



9.2 Let $\triangle ABC$ be a triangle, with M a point on $\overline{AC}$. Then

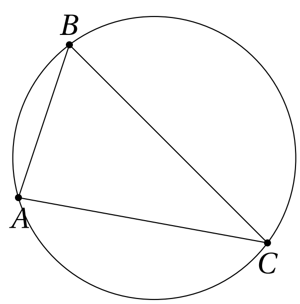
$$\frac{\text{Area}(\triangle ABM)}{\text{Area}(\triangle BMC)} = \frac{AM}{MC}. \tag{9.2}$$

That is, the ratio into which cevian $\overline{BM}$ divides the length of $\overline{AC}$ is the same as the ratio into which it divides the area of $\triangle ABC$ into smaller triangles $\triangle ABM$ and $\triangle BMC$.

Remark. Note that the right-hand side of (9.2) is a ratio of the lengths cut by cevian $\overline{BM}$. Following the definition of mass point sum, this suggests that by assigning suitable masses to A and C, we can balance $\overline{AC}$ at M. It follows that questions about *ratios of lengths* are equivalent to questions about *ratios of areas*, so both can be analyzed in the context of *mass point geometry*.

9.3 *The Law of Sines:* Let $\triangle ABC$ be a triangle, where R is the radius of it circumcircle. Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$



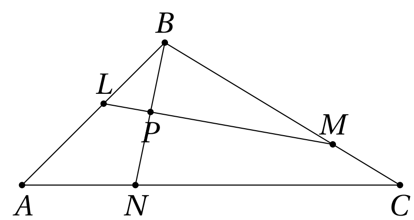
Hint: Let O be the center of this circle, and extend segment $\overline{BO}$ to a diameter of the circle. What is the measure of $\angle BAD$?

Remark. To reiterate, proving this theorem need not use mass point geometry, though we will find The Law of Sines useful in future problems using mass point geometry. (See Exercise #10.3(a) in particular.)

Remark. For earlier exercises with mass point geometry, we assigned masses when we had information about the ratios of lengths of sides. If we have information about angles—or, equivalently, about their trigonometric values—then we may use that information to assign masses to points in such a way that relevant mass point sums balance at the endpoint of a cevian. See Exercises #10.3 and #10.4 for examples.

10 Exercises

10.1 Consider $\triangle ABC$, with cevian $\overline{BN}$ and transversal $\overline{LM}$ that intersect in a common point P, as below.[4] If $AL/LB = 4/3$, $BM/MC = 5/2$, and $CN/NA = 7/3$, then compute the ratios LP/PM and BP/PN.



Hint: Using Strategy 7.4.4, split a mass point at B. How would you assign two masses m and n to B to produce the mass point sum $mB + nB$ at B?

10.2 *Varignon's Theorem:* Let A, B, C, and D be the vertices in the plane of a (nondegenerate) quadrilateral. Let K, L, M, and N be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Prove that $KLMN$ is a parallelogram.

Note: We are not assuming that $ABCD$ is a *convex* quadrilateral.

10.3 Let $\triangle ABC$ be a triangle with cevian $\overline{BM}$ bisecting $\angle ABC$.

[4] This was Exercise #5.1 from the last session.

(a) Show that

$$\frac{AM}{MC} = \frac{\sin C}{\sin A}.$$

Equivalently, show that $AM \sin A = MC \sin C$.

(b) Assume $\sin A = 3/5$ and $\sin C = 7/25$. Consider the median $\overline{AN}$, which bisects $\overline{BC}$, and let P denote the intersection of $\overline{BM}$ and $\overline{AN}$. Compute AP/PN and BP/PM.

10.4 Let $\triangle ABC$ be a triangle. Show that the angle bisectors of $\triangle ABC$ are concurrent. That is, show that the angle bisectors intersect in a single point. (Draw a picture!)

10.5 Let $\triangle ABC$ be an acute triangle. Prove that the altitudes—that is, the cevians from a vertex to the opposite site that are parallel to that side—of $\triangle ABC$ are concurrent. See Figure 10.1 for an example.

10.6 *Ceva's Theorem:* Let $\triangle ABC$ be a triangle with cevians $\overline{AL}$, $\overline{BM}$, and $\overline{CN}$. Then the cevians are concurrent if and only if

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1. \tag{10.1}$$

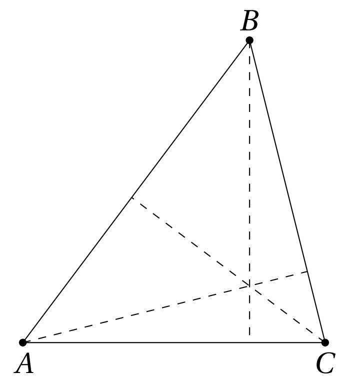
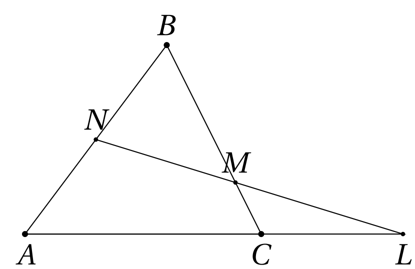


Figure 10.1: The altitudes of an acute triangle are concurrent.

Remark. Note that we could have solved Exercises #10.4 and #10.5 by first proving Ceva's Theorem, then showing that the product given above equals 1 in each case.

10.7 *Menelaus' Theorem:* Let $\triangle ABC$ be a triangle, L a point on the line given by $\overline{AC}$ as shown below, N any point in $\overline{AB}$, and M any point on $\overline{BC}$.



Then L, M, and N are collinear if and only if

$$\frac{AN}{NB} \cdot \frac{BM}{MC} \cdot \frac{CL}{LA} = 1.$$

10.8 In $\triangle ABC$, if cevians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ meet at P, then

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

11 Exercises from Mathematics Competitions

11.1 (**ARML 1989**) In $\triangle ABC$, angle bisectors $\overline{AD}$ and $\overline{BE}$ intersect at P. If the sides of the triangle are $a = 3$, $b = 5$ and $c = 7$, with $BP = x$ and $PE = y$, then compute the ratio x/y.

11.2 (**AHSME 1975**) In $\triangle ABC$, M is the midpoint of side $\overline{BC}$, $AB = 12$, and $AC = 16$. Points E and F are taken on $\overline{AC}$ and $\overline{AB}$, respectively, and lines $\overline{EF}$ and $\overline{AM}$ intersect at G. If $AE = 2AF$, find EG/GF.

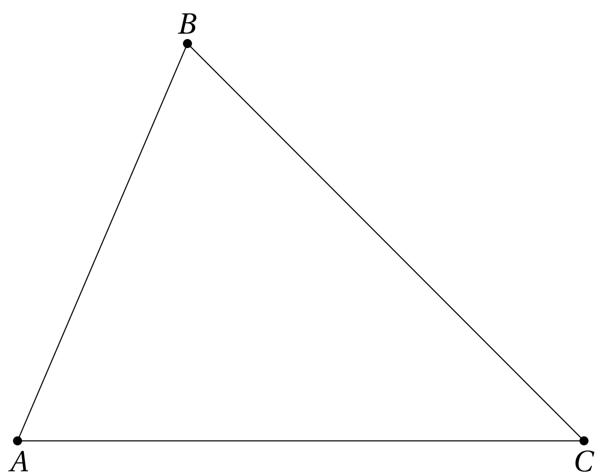
11.3 (**ARML 1992**) In $\triangle ABC$, points D and E are on $\overline{AB}$ and $\overline{AC}$, respectively. The angle bisector of $\angle A$ intersetcs $\overline{DE}$ at F and $\overline{BC}$ at T. If $AD = 1$, $DB = 3$, $AE = 2$, and $EC = 4$, compute the ratio AF/AT.

11.4 (**AIME 1988**) Let P be an interior point of $\triangle ABC$, and extend lines from the vertices through P to the opposite sides. Let $AP = a$, $BP = 5$, $CP = c$, and let the extensions from P to the opposite sides all have length D. If $a+b+c=43$ and $d=3$, then find abc. (Cf. [4].)

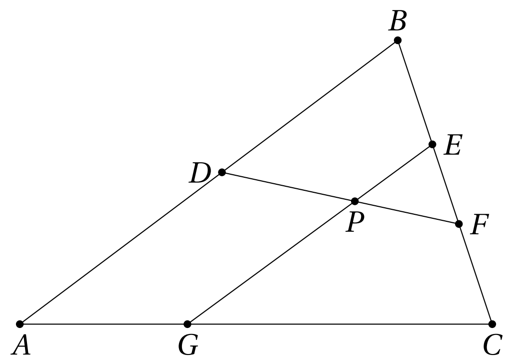
12 Further Additional Exercises

12.1 In $\triangle ABC$, points D and E are on $\overline{AB}$ and $\overline{BC}$, respectively/ Let P denote the point of intersection of $\overline{AE}$ and $\overline{CD}$.

Set $X := \text{Area}(\triangle ADP)$, $Y := \text{Area}(\triangle APC)$, and $Z := \text{Area}(\triangle CEP)$. If $X/Y = 3/7$ and $Y/Z = 4$, then compute the ratios AP/PE and CP/PD. Can you also compute the ratios AD/DB and BE/EC?



12.2 **Challenging:** Consider $\triangle ABC$, where D lies on $\overline{AB}$, E and F lie on $\overline{BC}$, and G likes on $\overline{AC}$. The transversals $\overline{DF}$ and $\overline{EG}$ intersect in a common point P, as below. If $AD/DB = 4/3$, $BE/EC = 5/2$, $BF/FC = r$, and $CG/GA = 7/3$, then compute the ratios DP/PF and BP/PG.



Hint: Consider a mass point geometry strategy with *two* split masses. (Where?) Select masses for A, B, and C with relevant balancing at D, and G, and so that the overall system is balanced at P. What would this mean regarding balancing at E and F?

Note: Proceeding with this method may require—or at least benefit from—some experience in linear algebra, enough to solve systems of linear equations.

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