An Introduction to Mass Point Geometry, Part 1 of 2

Abstract

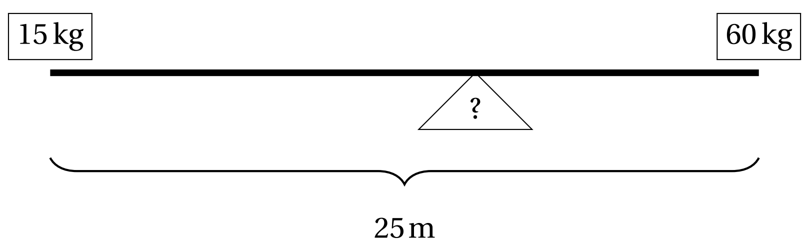
In this session, we shall introduce *mass point geometry*. Motivation comes from physics, generalizing the notion of the center of mass. Mass point geometry techniques can be powerful, offering much simpler ways to solve a number of problems from classical geometry, both in the plane and in higher dimensions. Our approach is modeled on that of [12] its subsequent adaptation in [10].

Background needed: We assume basic familiarity with geometry, including describing points in $\mathbb{R}^n$ as n-tuples of the form $(x_1, x_2, \ldots, x_n)$, and making algebraic computations associated with such a description of points.

0 Warmup

Exercises:

0.1 Consider a rigid bar, like a seesaw, of length 25 m and negligible mass. We place masses 15 kg and 60 kg at each respective end:



In order for the system, including these weights at those positions, to balance parallel to the ground, where should we place the fulcrum? (Don't worry about a formal proof for *why* your selection for the location of such a pivot point would work.)

1 Preliminaries: Definitions and Notation

We begin with some notation and definitions:

Definition 1.1. Let P be a point, and $m > 0$ a real number. Then a *mass point* is an expression denoted mP. Two mass points mP and nQ are defined to be equal if and only if $m = n$ and $P = Q$. We shall often simply write "P" as shorthand for the mass point $1P$, too, provided the context is clear that this represents a mass point and not simply a point.

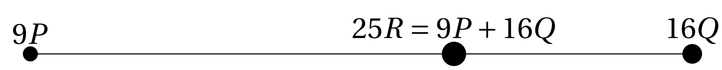
Definition 1.2. Let mP and nQ be two mass points. The *mass point sum* of mP and nQ, denoted $mP + nQ$, is defined as follows:

- If $P = Q$, then $mP + nQ := (m+n)P$.
- If $P \neq Q$, then $mP + nQ$ is defined to be mass point of mass $m+n$ at the unique point R on the line segment $\overline{PQ}$ that lies $n/(m+n)$ of the distance PQ from P to Q. That is,

$$\frac{PR}{RQ} = \frac{n}{m}.$$

- If mP is a mass point and $a > 0$, then we define $a(mP) := (am)P$.

Intuitively, the point R is the center of mass for the set $\{mP, nQ\}$ of mass points. If you prefer, imagine, as in Exercise 0.1, a balance beam with the given masses at the respective points. Its balancing point is at R, and the mass at point R is $m+n$, the sum of the masses of the other given points. See the example below:



The mass point sum $mP + nQ$ then represents a kind of summary of the entire system, encoding both *total mass* and the location of the *center of mass*. Namely, the mass point $mP + nQ$ sum has mass $m+n$, the sum of the two given masses. Further, its location is at the center of mass of the two given masses at their respective locations.

Note in particular that the mass point sum is *closer* to the point with *larger* mass. When $m = n$, so that the masses are equal, then the mass point $mP + nQ = mP + mQ$ lies at the midpoint of the segment $\overline{PQ}$.

Proposition 1.3. *Let ℓO, mP, and nQ be any mass points, and assume $a > 0$ Then*

- *Mass point addition is commutative: $mP + nQ = nQ + mP$.*
- *Mass point addition is associative: $\ell O + (mP + nQ) = (\ell O + mP) + nQ$*
- *Mass point addition is distributive: $a(mP + nQ) = amP + anQ$.*

All these properties can (and should!) be justified, of course. (See, for example, Exercise #5.4.) For now, though, let's accept these provisionally and use the properties of mass points and their sums to solve certain exercises.

2 Working with Mass Point Sums

Exercises: Throughout, your solutions need not be drawn precisely to scale. So long as you can explain what's going on correctly, and generally explain where the point should appear, that will suffice.

2.1 Draw the following mass point sum $mP + nQ$:

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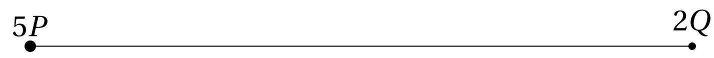
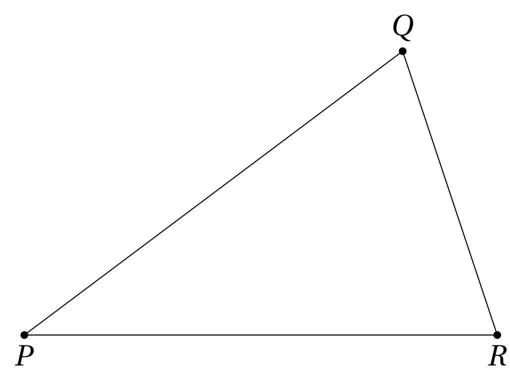
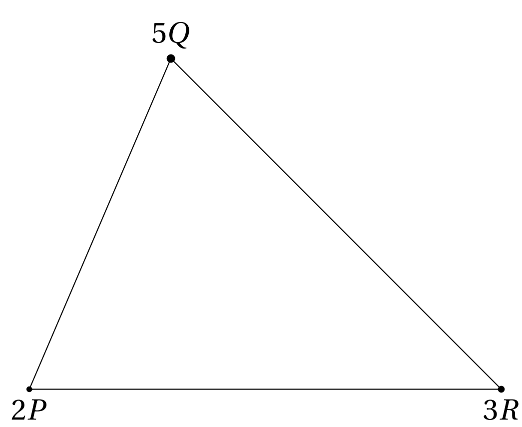


2.5 Draw the mass point sum $1P + 1Q + 1R$ indicated below:



2.6 Draw the mass point sum $2P + 5Q + 3R$ indicated below:



3 Basic Strategies with Mass Points

Strategy 3.1. *The following are strategies for using mass points:*

(a) *Assume that $\overline{PQ}$ is a line segment containing the point R. Then if we know $PR/RQ = n/m$, assign masses m and n to P and Q respectively so that $mP + nQ = (m+n)R$. Further, we can scale this by a positive constant k to have $k(m+n)R = kmP + kmQ$, as well.*

Recall that the numerator in this ratio is assigned to Q and the denominator is assigned to P, so that the larger *mass is assigned to which of P or Q is* closer *to R.*

(b) *Given mass points mP and nQ, use mass point operations to determine the location of R on $\overline{PQ}$ such that $mP + nQ = (m+n)R$.*

(c) *Let P, Q, and R be points. If for some masses m and n we have that $mP+nQ = (m+n)R$, then R must lie on segment $\overline{PQ}$.*

That is, if we can show that a particular point R is the location of a mass point sum of the form $mP+nQ$, then we can conclude that R lies on $\overline{PQ}$. This will be especially useful in showing two lines intersect in a particular *point, or that three lines are collinear.*

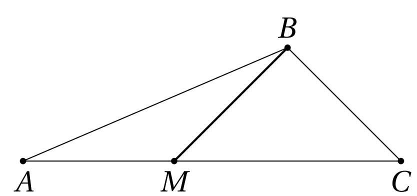
(d) *We can "split" a single mass point, representing it as the sum of two mass points corresponding to the same underlying point.*

For example, a given mass point $(m+n)P$ can be rewritten in the equivalent form $mP + nP$. (This will be easier to appreciate with examples; see Exercise #5.1 below.

In particular, Strategies 3.1(b)–3.1(c) are often useful to show that that *three* line segments intersect in a common point. Strategy 3.1(d) is invaluable when our desired assignment of mass points would otherwise be inconsistent with the given hypotheses.

4 Cevians and Mass Points

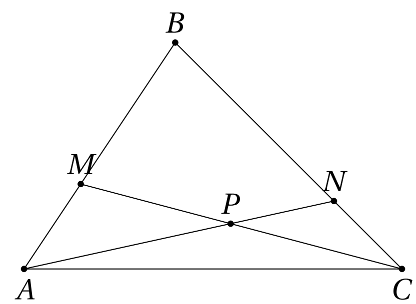
Definition. Let $\triangle ABC$ be any triangle. A *cevian*[1] is a line segment connecting one of the vertices of the triangle with any point (excluding the endpoints) of the opposite side. For example, $\overline{BM}$ below is a cevian of $\triangle ABC$.



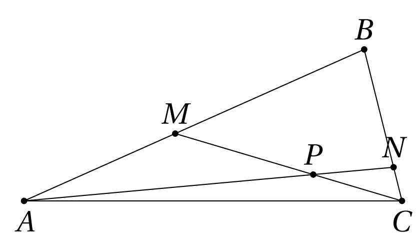
[1] This term named for Italian mathematician Giovanni Ceva. Following Italian pronunciation rules, it would therefore be pronouced "CHAY-vee-uhn".

4.1 Consider a triangle $\triangle ABC$. A *median* of a triangle is a line segment whose endpoints are one of the triangle's vertices and the midpoint of the opposite side. Prove that the three medians of any triangle intersect in a common point, called the *centroid* of a triangle. Further, prove that the medians divide each other in the ratio 2-to-1, where the point of intersection lies farther from each vertex than from the opposite site.

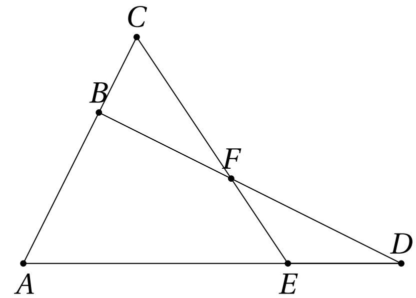
4.2 Consider $\triangle ABC$, with cevians $\overline{AN}$ and $\overline{CM}$ that intersect in a common point P, as below. If $AM/MB = 3/5$ and $BN/NC = 7/3$. Compute the ratios AP/PN and CP/PM.



4.3 Consider $\triangle ABC$, with cevians $\overline{AN}$ and $\overline{CM}$ that intersect in a common point P, as below. If $AM/MB = 4/5$ and $BN/NC = 7/2$. Compute the ratios AP/PN and CP/PM.

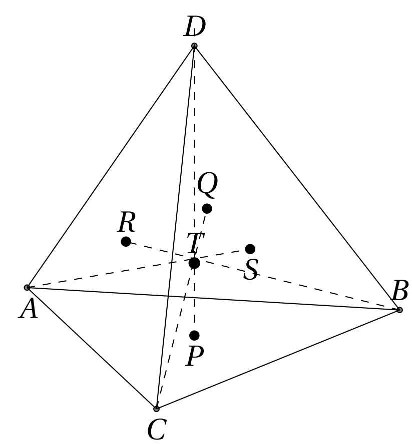


4.4 Consider the diagram below. If $AB/BC = 2$ and $AE/ED = 7/3$, then compute BF/FD and CF/FE.



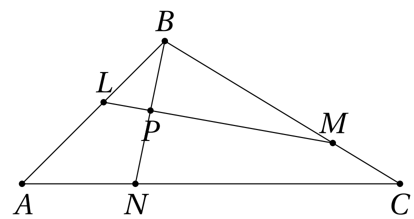
4.5 Consider a tetrahedron $ABCD$ in space. Let P, Q, R, and S be, respectively, the centroids of $\triangle ABC$, $\triangle ABD$, $\triangle ACD$, and $\triangle BCD$. (See Exercise #4.1 for the definition of the centroid of a triangle.) Prove that the line segments $\overline{AS}$, $\overline{BR}$, $\overline{CQ}$, and $\overline{DP}$ all intersect in a common point T. What are the ratios AT/TS, BT/TR, CT/TQ, and DT/TP?

Can you generalize this result to polyhedra in dimension 4 or higher?



5 Additional Exercises

5.1 Consider $\triangle ABC$, with cevian $\overline{BN}$ and transversal $\overline{LM}$ that intersect in a common point P, as below. If $AL/LB = 4/3$, $BM/MC = 5/2$, and $CN/NA = 7/3$, then compute the ratios LP/PM and BP/PN.



5.2 Given mass points mP and nQ, how might you define the mass point difference $mP - nQ$? Under what conditions would $mP - nQ$ exist?

5.3 Say that mP and nQ are mass points, where in terms of Cartesian coordinates, $P := (x_1, y_1)$ and $Q := (x_2, y_2)$. What are the Cartesian coordinates of the point R, where $mP + nQ = (m+n)R$? What if we are in three-dimensional space and the points have coordinates given by $P := (x_1, y_1, z_1)$, $Q := (x_2, y_2, z_2)$? Can this generalize to higher dimensions?

Hint: If the general case seems too abstract, consider the following special case. Given distinct points $P := (x_1, y_1, z_1)$ and $Q := (x_2, y_2, z_2)$, what are the coordinates of the midpoint M of the segment $\overline{PQ}$? What about the point R on $\overline{PQ}$ that lies one-fourth of the way from P to Q on this segment?

5.4 Earlier, we asked you to use Proposition 1.3 above without yet justifying it. Now, prove each of its three claims.

References

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