Combinatorics, Part 2[1]

Warm-up

1. How many ways are there to make a playlist of 5 songs out of your favorite 12 songs?

   Answer: $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$

2. How many ways are there to pick 5 ingredients to put in a stir fry, out of 12 ingredients to choose from?

   Answer: $\dfrac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

Permutations and Combinations

For integers n and r with $0 \leq r \leq n$,
Permutations:
Permutations: The number of ways to arrange n different objects in a row is

$$n! = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1$$

The number of ways to arrange r objects in a row, when there are n distinct objects to choose from, is

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

Combinations:
The number of ways to choose r objects out of a collection of n objects is

$$C(n,r) = \binom{n}{r} = \frac{n \cdot (n-1) \cdot (n-2) \ldots (n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$$

3. In how many ways can you choose team of 6 volunteers to staff a math festival from an applicant pool of 11 undergraduates and 17 grad students? What if the team has to consist of 3 undergrads and 3 grad students?

   Answer: $\binom{28}{6}$, $\binom{11}{3} \cdot \binom{17}{3}$

4. How many ways can you make a pizza with 3 different toppings if there are 8 toppings to choose from? What if the 3 toppings don't all have to be different? For example, you could have mushroom and double pepperoni as one option.

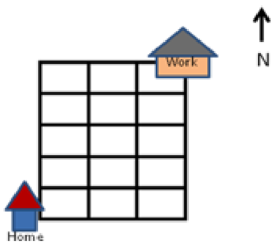
   Answer: $\binom{8}{3} = 56$. *If they don't have to be different, then* $\binom{8}{3} + \binom{8}{2} \cdot 2 + 8 = 120$

[1] Many of these problems are from *Mathematical Circles (Russian Experience). Others are from Paul Zeitz.*

5. If you flip a coin 10 times, how many of the outcomes have exactly 4 heads and 6 tails?

   Answer: $\binom{10}{4}$

6. How many different routes are there from home to work, only traveling north and east on streets?



   Answer: $\binom{8}{3}$

7. A school has 100 students enrolled in fifth grade and 5 fifth grade teachers, each with their own classroom.

   (a) In how many ways can student assignments be made if each classroom holds 20 students?

   Answer: $\binom{100}{20}\binom{80}{20}\binom{60}{20}\binom{40}{20}\binom{20}{20} = \dfrac{100!}{(20!)^5}$

   (b) Nora and Sophia are hoping to be in the same class in fifth grade. How many ways can student assignments be made in which this happens?

   Answer: $5 \cdot \binom{98}{18}\binom{80}{20}\binom{60}{20}\binom{40}{20}\binom{20}{20} = \dfrac{5 \cdot 98!}{18! \cdot (20!)^4}$

8. There are 5 time slots in a summer camp schedule, and one activity must be put into each slot to form a schedule. Each schedule must contain 3 sports activities, out of 7 to choose from, and 2 arts and crafts activities, out of 5 to choose from. How many schedules are possible?

   Answer: $\binom{7}{3}\binom{5}{2} \cdot 5!$

9. How many ways can you choose a team of 8 or more people from 20 people, where each team must have at least one person and the team must have a designated captain?

   Answer: $\binom{20}{8} \cdot 8$ *or* $20 \cdot \binom{19}{7}$

10. How many ways can you choose a team of 2 or more people from 11 people, where each team must have at least one person and the team must have a designated captain?

    Answer: $11 \cdot (2^{10} - 1)$

11. How many ways are there to arrange the letters in the word HOGWARTS? The letters in the word VOLDEMORT? The letters in the word ALOHOMORA? The letters in the words AVADAKEDAVRA?

    Answer: HOGWARTS $8!$, *VOLDEMORT* $\dfrac{9!}{2}$, *ALOHOMORA* $\dfrac{9!}{3!}$

12. How many ways are there to distribute 10 doggie biscuits among 7 dogs? The biscuits are indistinguishable, but the dogs are distinguishable.

    Answer: $\binom{16}{6}$

Stars and Bars Rule
The number of ways to distribute n indistinguishable objects into k distinguishable boxes is given by

Anagram Rule
The number of ways to arrange n objects, when there are r_i indistinguishable objects of type 1, r_2 indistin objects of Type 2, ... and r_k indistinguishable objects of Type k, is given by:
.

13. You have 2 Reece's pieces, 4 mini bags of M&M's, and 1 sour candy left from Halloween. You are going to eat one candy per day until the candy is gone. How many different ways can you do this?

    Answer: $\frac{7!}{2!}4!$

14. There are 10 questions on a math final exam. How many ways are there to assign point values to the problems if the sum of the scores is 100 and each question is worth at least 5 points?

15. How many ways can n books be placed on k distinguishable shelves

    (a) if the books are indistinguishable copies of the same title?

    (b) if no two books are the same, and the positions of the books on the shelves matters?

16. How many ways are there to represent the number 12 as a sum of

    (a) 5 non-negative integers?
    Answer: $\binom{16}{4}$

    (b) 5 positive integers? *Answer:* $\binom{11}{4}$

17. You have a stack of 45 identical blank cards. On each one, you draw a circle, a star, or a square. How many different stacks of 45 cards are possible? Assume all circles look alike, all squares look alike, and all stars look alike. Two stacks that have different orders of cards are considered different.

    Answer: 3^{45}

18. Same problem, but this time you are dumping all the cards into a bag. How many different bags of 45 cards are possible?

    Answer: $\binom{47}{2}$

19. How many ways are there to rearrange the letters in the word "FLAMINGO" so that the vowels will be in alphabetical order and so will the consonants? For example, FAGILMON (A - I - O, F - G - L - M - N).

Challenge

20. How many ways are there to travel in xyzw space from the origin $(0, 0, 0, 0)$ to the point $(4, 3, 5, 4)$ by taking steps of one unit in the positive x, positive y, positive z, or positive w direction?

21. What is the largest number of triangles you can make by drawing 7 lines in the plane? The triangles may overlap or contain each other.

    Answer: $\binom{7}{3}$

22. There are three rooms in a dormitory: one single, one double, and one for four students. How many ways are there to house seven students in these rooms?

23. You flip a coin 10 times. Of all the possible outcomes, how many have exactly 5 heads in a row? For example, we would not count HHHHHHHTTT (too many consecutive heads), but we would count TTTHHHHHTT and HHTTHHHHHT.

    Answer: They must have a T before and after the run of 5 heads, unless the run is at the beginning or the end. If it is at the beginning, 6 spots are fixed, so 4 spots are flexible, so $2^4 = 16$ *options, same if it is at the end, and otherwise 7 are fixed so* $2^3 = 8$ *options, but there are 4 ways to do this depending if there are 0, 1, 2, or 3 coins before the string of 7. This totals to* $16 + 16 + 4 \cdot 8 = 64$

24. 10 cats and 9 dogs sit in a row of 19 cushions. How many ways can this be done if

    (a) All the dogs sit next to each other and all the cats sit next to each other.

    Answer: $10! \times 9! \times 2$

    (b) The pets sit so that each dog has only cat neighbors and each cat has only dog neighbors.

    Answer: $10! \times 9!$

25. In a traditional village, there are 10 young men and 10 young women. The village matchmaker arranges all the marriages.

    (a) In how many ways can they pair off the 20 young people, assuming a marriage has to be between a man and a woman?

    Answer: $6!$

    (b) In how many ways can they pair off the 20 young people if male-male couples, male-female couples, and female-female couples are all allowed?

    Answer: $19 \cdot 17 \cdot 15 \cdot 13 \cdot 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$

26. In how many ways can a photographer at a wedding of a bride and groom arrange 8 people in a row, including the bride and groom, if:

    (a) the bride must be next to the groom?

    (b) the bride must NOT be next to the groom?

    (c) the bride is positioned somewhere to the left of the groom?

Poker hands

27. How many ways are there to choose 6 cards from a complete deck of 52 cards in such a way that all four suits will be present?

    Answer: $4 \cdot \binom{13}{3} \cdot 13^3 + \binom{4}{2} \cdot \binom{13}{3}^2 \cdot 13^2$

28. Consider the following poker hands:

    (a) Royal flush: A, K, Q, J, 10, all the same suit.

    (b) Straight flush: Five cards in a sequence, all in the same suit.

    (c) Four of a kind: All four cards of the same rank.

    (d) Full house: Three of a kind with a pair.

    (e) Flush: Any five cards of the same suit, but not in a sequence.

    (f) Straight: Five cards in a sequence, but not of the same suit.

    (g) Three of a kind: Three cards of the same rank.

    (h) Two pair: Two different pairs.

    (i) Pair: Two cards of the same rank.

    Find the probability of each kind of hand.