Chapel Hill Math Circle
Session 9 – February 3, 2024: Integer Partitions and Compositions
Beginners' Group (grades 1-3), 10:30-11:30a
Mr. Barman – dilip@trianglemathinstitute.com

Supplies needed: 5-10 large and 5-10 smaller coins or tokens; colored rectangles that need to be cut out are included here.

Happy February! Those of you who were here last time learned a bit about breaking up numbers. We call this idea "partitions". We will review a bit of what we did so that you can enjoy today even if you weren't here last time, and look at some new ways of partitioning numbers. We'll even prove an interesting result! Are you ready to have some fun?

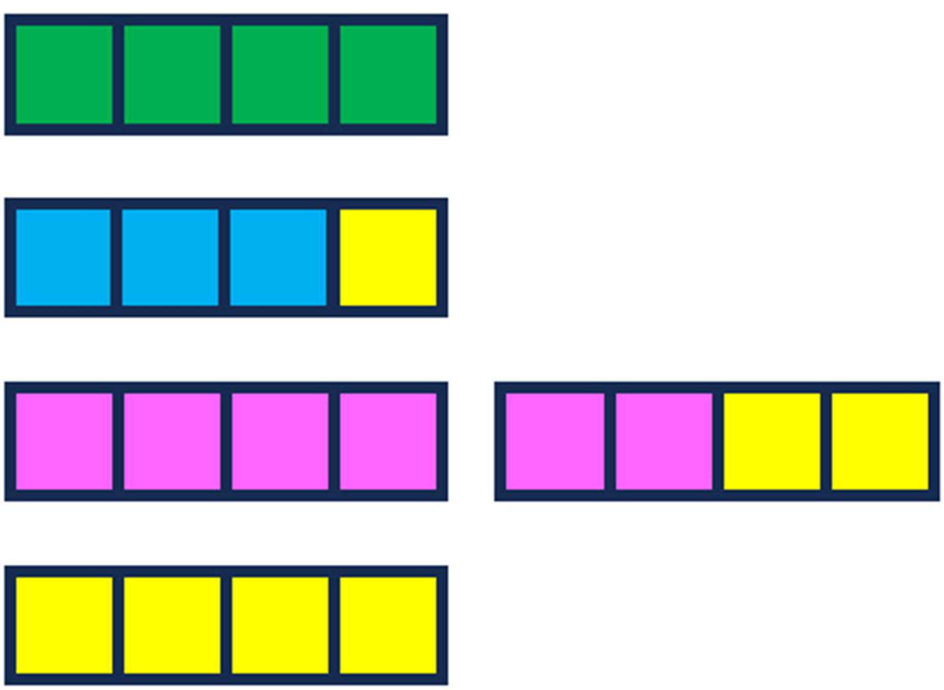
Last time we broke up numbers

Let's warmup and try this – how many ways are there to break up 4? We can have 2+2. 3+1 works – as does 1+3. Let's agree that order doesn't matter so 3+1 is the same as 1+3. How about 1+1+1+1? And just 4 (4+0)?

Let's improve this and be organized. Let's start with the biggest numbers first:

- 4 (1 way)
- 3+1 (1 way)
- 2+2, 2+1+1 (we missed that before!) (2 ways)
- 1+1+1+1 (1 way; total of 1+1+2+1 = 5 ways)

Now it's your turn. **Can you show how to break up 5, 6, and 7?** You can use the rectangles in this packet; for example, here are the ways to make 4 with 4 (green), 3 (blue), 2 (pink), and 1 (yellow):



Did you get this for 5?

- 5 (1 way)
- 4+1 (1 way)
- 3+2, 3+1+1 (2 ways)
- 2+2+1, 2+1+1+1 (2 ways)
- 5 1s (1 way; total of 1+1+2+2+1 = **7 ways**)

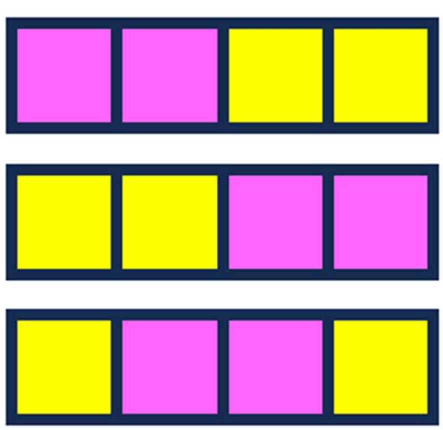
Here's a summary table of what I got for the first 10 cases.

Number	Ways to break by largest number	Number of ways
0	0(1)	1
1	1(1)	1
2	2(1), (1)	2
3	3(1), 2(1), 1(1)	3
4	4 (1), 3(1), 2(2), 1(1)	4+1 = 5
5	5(1), 4(1), 3(2), 2(2), 1(1)	4+2+1 = 7
6	6(1), 5(1), 4(2), 3(3), 2(3), 1(1)	4+3+3+1 = 11
7	7(1), 6(1), 5(2), 4(3), 3(4), 2(3), 1(1)	4+3+4+3+1 = 15
8	8(1), 7(1), 6(2), 5(3), 4(5), 3(5), 2(4), 1(1)	4+3+5+5+4+1 = 22
9	9(1), 8(1), 7(2), 6(3), 5(5), 4(6), 3(7), 2(4), 1(1)	4+3+5+6+7+4+1 = 30
10	10(1), 9(1), 8(2), 7(3), 6(5), 5(7), 4(9), 3(8), 2(5), 1(1)	4+3+5+7+9+8+5+1 = 42

Do you see a pattern? For 4 and above, **why does the total always start with 4?** If we have time today or maybe in the future we can try to find more patterns. But for now let's move on to …

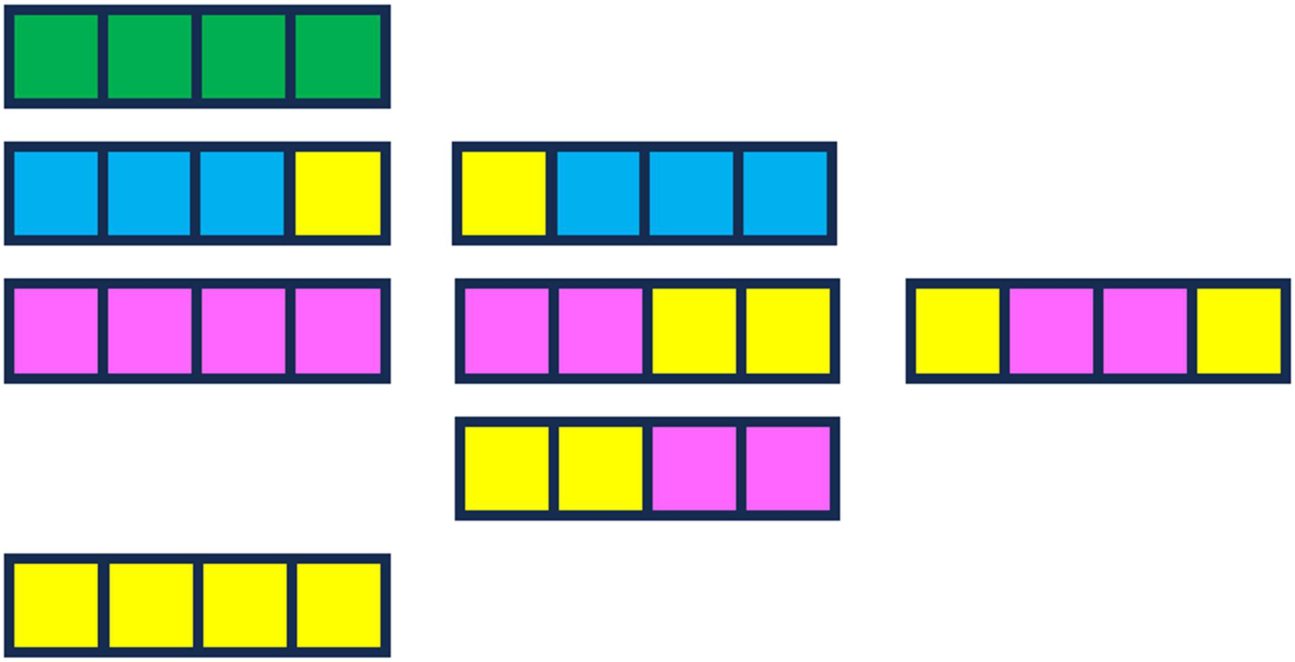
Let's look at breaking down numbers where order matters

Think of making a train where order matters. If we make a train of length 0, 1, or 2, there is no difference between order mattering and order not mattering.



But what if we have a train that is at least 3 long. Aren't these different trains to the right? And don't we have more than 5 ways to make 4? Instead of 2+1+1 being the only way to make 4 with a 2, don't we also have 1+1+2 and even 1+2+1? Can you figure out **how many ways there are to make 4 when order matters?**

When order matters, there are actually 8 ways to make 4, aren't there?



Mathematicians call breaking numbers when order doesn't matter **partitions**, and **compositions** when order does matter.

So far this is what we've found.

	Order does NOT matter (Partitions)		Order DOES matter (Compositions)	
Number	**Ways to break**	**Number of ways**	**Ways to break**	**Number of ways**
0	0	1	0	1
1	1	1	1	1
2	2; 1+1	2	2; 1+1	2
3	3; 2+1; 1+1+1	3	3; 2+1; 1+2; 1+1+1	4
4	4; 3+1; 2+2, 2+1+1; 1+1+1+1 OR: 4; 3+1; 2 2s, 2 and 2 1s; 4 1s	5	4; 3+1, 1+3; 2+2, 2+1+1, 1+2+1, 1+1+2; 4 1s	8
5	5; 4+1; 3+2, 3+1+1; 2+2+1, 2+1+1+1; 5 1s	7		

Please fill in the row for 5; **how many and what are the ways to make 5 when order matters?**

Did you get this for 5?

1 5: 5
1 4: 4+1, 1+4
1 3: 3+2, 2+3; 3+1+1, 1+1+3, 1+3+1;
2 2s: 2+2+1, 1+2+2; 2+1+2
1 2: [we've already counted 2+3]; 2+1+1+1, 1+1+1+2; 1+2+1+1, 1+1+2+1;
1s: [we've already counted all with 1 1, 21s, 3 1s, and 4 1s] 5 1s

The count is 1 + 2 + 5 + 3 + 4 + 1. This way of organizing was good with partitions but let's organize ourselves a little differently. How about if we countdown with the first number and then always start with the biggest number:

5;
4+1;
3+2, 3+1+1;
2+3, 2+2+1, 2+1+2, 2+1+1+1;
1+4, 1+3+1, 1+2+2, 1+2+1+1, 1+1+3, 1+1+2+1, 1+1+1+2, 1+1+1+1+1

The count is 1 + 1 + 2 + 4 + 8 = 16.

Do you have a guess for how many compositions 6 has?

There was **1** way to make 1, **2** ways to make 2, **4** ways to make 3, **8** ways to make 4, and **16** ways to make 5. **What is your prediction for the number of compositions of 6?**

Do you get a sense that we double each time?

Let's see if we can convince ourselves of this. You will need 5 big objects and 4 small objects. I've used coin images from usmint.gov.

Put the 5 big objects in a row with space in between. You can put your small object into any of the 4 spaces in between. The gaps are just separators.

Here I've chosen **0** dimes. This is just 1 unseparated group of 5, so this is the case of **5**.

Now I've chosen **1** dime. I can put that 1 dime in 4 places; here I put it in the first gap.

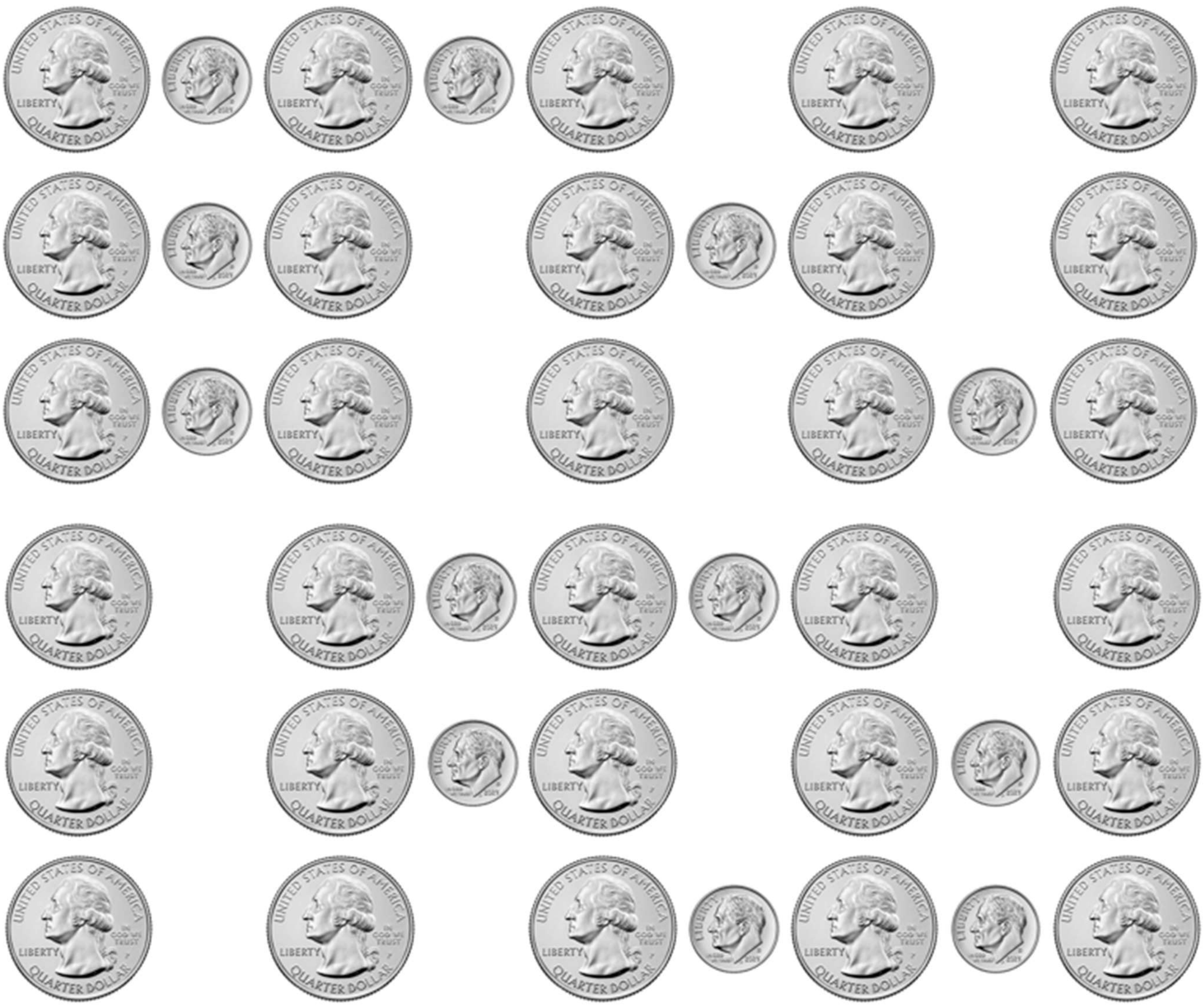
The dime is a separator; isn't this just 1 (the 1 quarter) plus 4 (the 4 quarters to the right), or **1+4**?

Let's put the dime in the second gap.

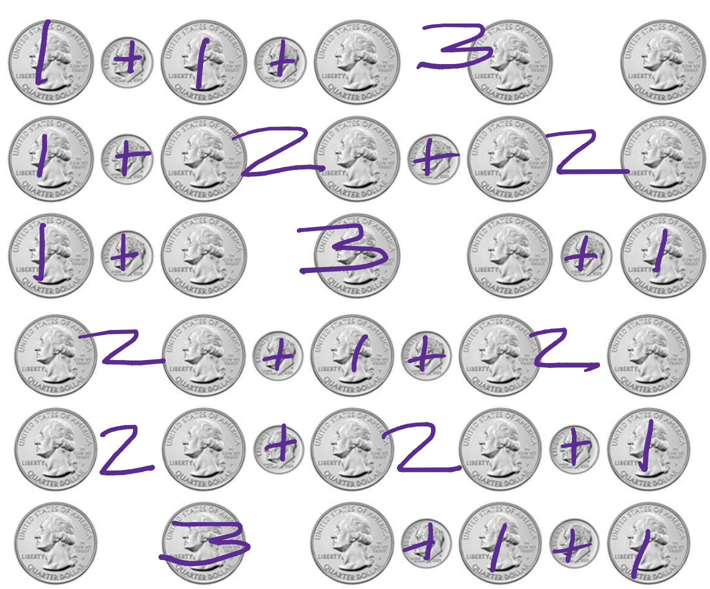
Isn't this like counting **2+3**? **Try it with your objects moving the small object to the other two gaps.** You should convince yourself that this corresponds to **3+2** and then **4+1**.

Now use two small coins. **Can you list all the possibilities of separating your 5 big objects with two small coin separators?** Use your coins or other objects to make all the combinations and then write down to what integer compositions these correspond.

Did you get something like this?



These correspond to these sums:



1 + 1 + 3
1 + 2 + 2
1 + 3 + 1
2 + 1 + 2
2 + 2 + 1
3 + 1 + 1

Please come up with all of the compositions of 5 using your two objects

I demonstrated 1 (1 way of placing 0 dimes) + 4 (ways of putting 1 dime in the 4 gaps) + 6 (ways of putting 2 dimes in 4 gaps). My prediction is that you will have 4 (ways of putting 3 dimes in 4 gaps – isn't this the same as picking one of the four gaps to be empty?) + 1 (ways of filling each of the 4 gaps with a dime). We then have 1 + 4 + 6 + 4 + 1 = 16 ways.

There's lots to think about!

- First of all, **can you convince yourself that our game with the objects gives us all of the compositions?** Jot down a few thoughts or pictures to convince yourself.
- Why does our construction show that the number of compositions keeps doubling? Think of it this way; for the case of 5, for example, we have 4 "gaps" each gap could have a divider or not. If there were 1 gap there would be two possibilities, nothing or a separator. When we have 2 gaps we can have nothing, separator in the first place only, separator in the second place only, or 2 separators: 4 possibilities. Each time our choices double, no?
- What pattern do you see in 1 + 4 + 6 + 4 + 1? **What sum adding up to 16+16 = 32 do you predict you will see in the case of compositions of 6?** We will look at this pattern next time when we look at Pascal's Triangle.
- **Complete as much of the table below as you wish.**

	Order does NOT matter (Partitions)		Order DOES matter (Compositions)	
Number	**Ways to break**	**Number of ways**	**Ways to break**	**Number of ways**
0	0	1	0	1
1	1	1	1	1
2	2; 1+1	2	2; 1+1	2
3	3; 2+1; 1+1+1	3	3; 2+1; 1+2; 1+1+1	4
4	4; 3+1; 2+2, 2+1+1; 1+1+1+1 OR: 4; 3+1; 2 2s, 2 and 2 1s; 4 1s	5	4; 3+1, 1+3; 2+2, 2+1+1, 1+2+1, 1+1+2; 4 1s	8
5	5; 4+1; 3+2, 3+1+1; 2+2+1, 2+1+1+1; 5 1s	7		16
6	6; 5+1; 4+2, 4+1+1; 3+3, 3+2+1, 3+1+1+1; 2+2+2, 2+2+1+1, 2+[4 1s]; 6 1s	11		
7		15		
8		22		
9		30		
10		42		

Activities for you to consider

Partitions and compositions are a rich area of math. Here are some activities for you to try with your family or by yourself.

- Tell a (real or made up) story about partitioning numbers. For example: *I had a heavy box of books to return to the library. The books weighed 20 pounds. I broke the 20 pounds into 3 piles, a big pile of 12 pounds for my dad, 1 pound for my little sister, and 7 pounds for me; it was easy for us three to carry the books from the box in the car into the library.*
- Tell another story related to compositions where order matters. For example: *A landscape company is loading 102 decorative bricks into a truck to use at a client's home. Some of the bricks come in packs of 20, some in packs of 5, and some are loose. The truck packs most efficiently if larger packs go before smaller ones. The truck driver sees 3 piles of 20 and dozens of dozens of packs of 5 and singles, so packs his truck with the 3 20-pound packs, 8 5-pound packs, and then 2 individual bricks.*
- Can you describe a way to break up numbers as partitions or compositions where it can help you in math? For example: *456 is made up of 400 and 50 and 6*. Maybe you have a way of breaking up numbers to make them easier to add or to break them into containers that hold, say, a dozen items. What if you have jewelry boxes that have 4 compartments each and you are able to fill 10 of them with jewels and have an 11th one with only 2 jewels? Doesn't that mean that you have 40 + 2 = 42 jewels since 10 4s is 40?

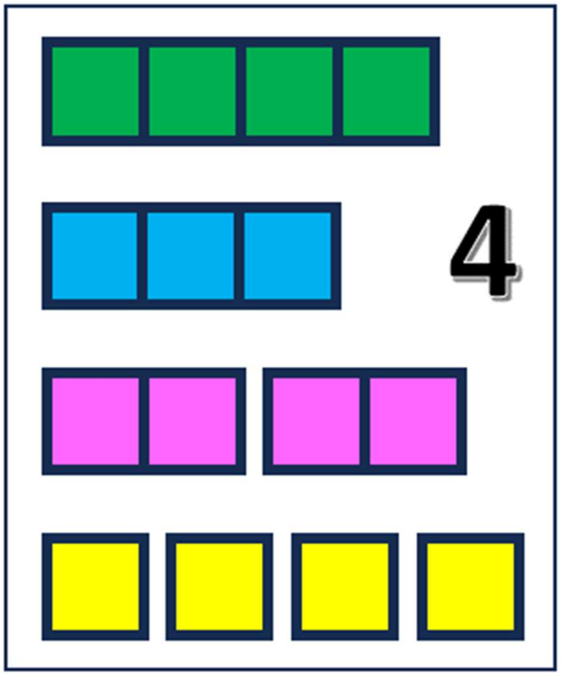
Have a good few weeks! See you after Valentine's Day!

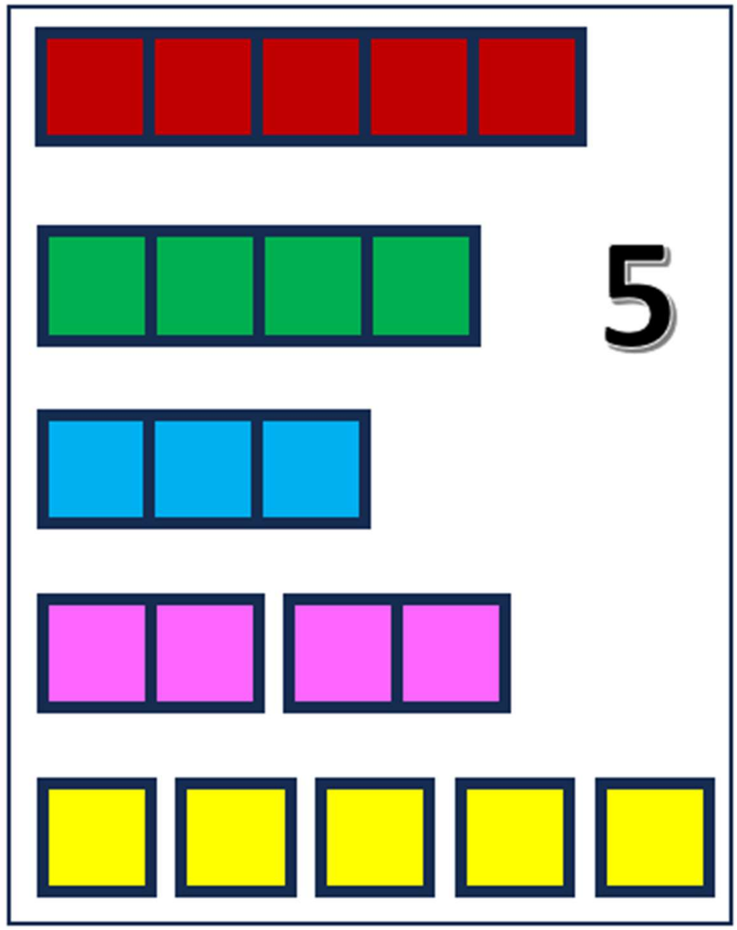
Have Fun!

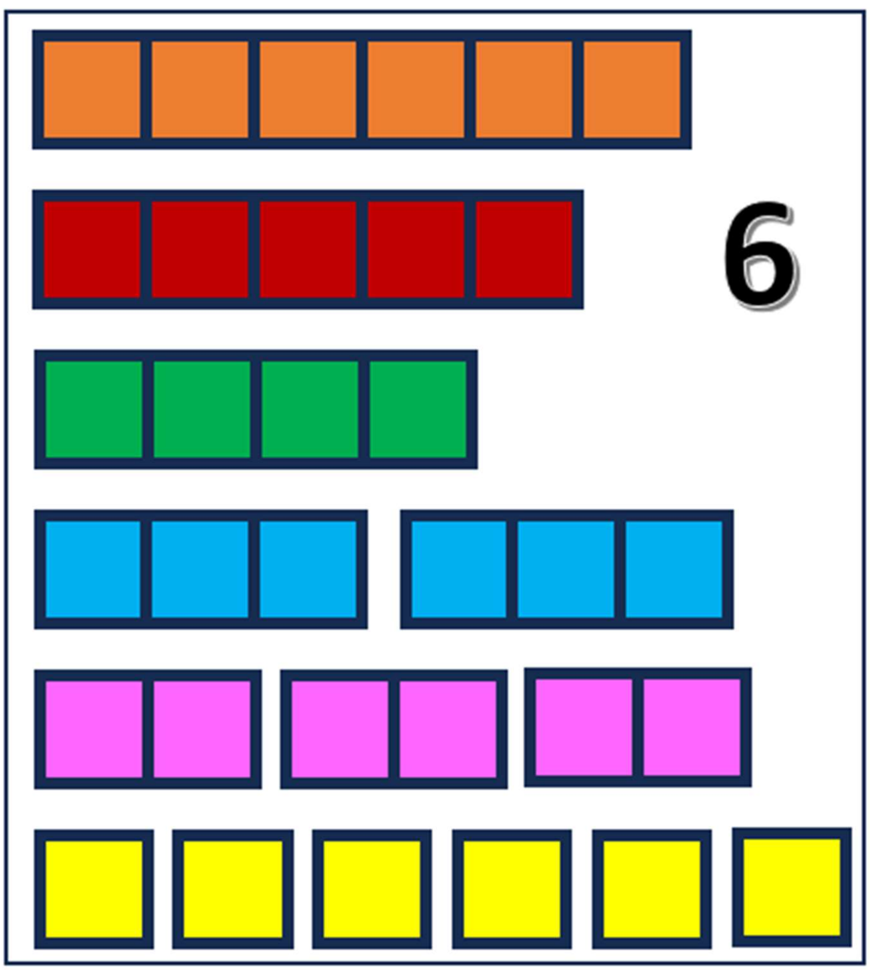
Mr. Barman

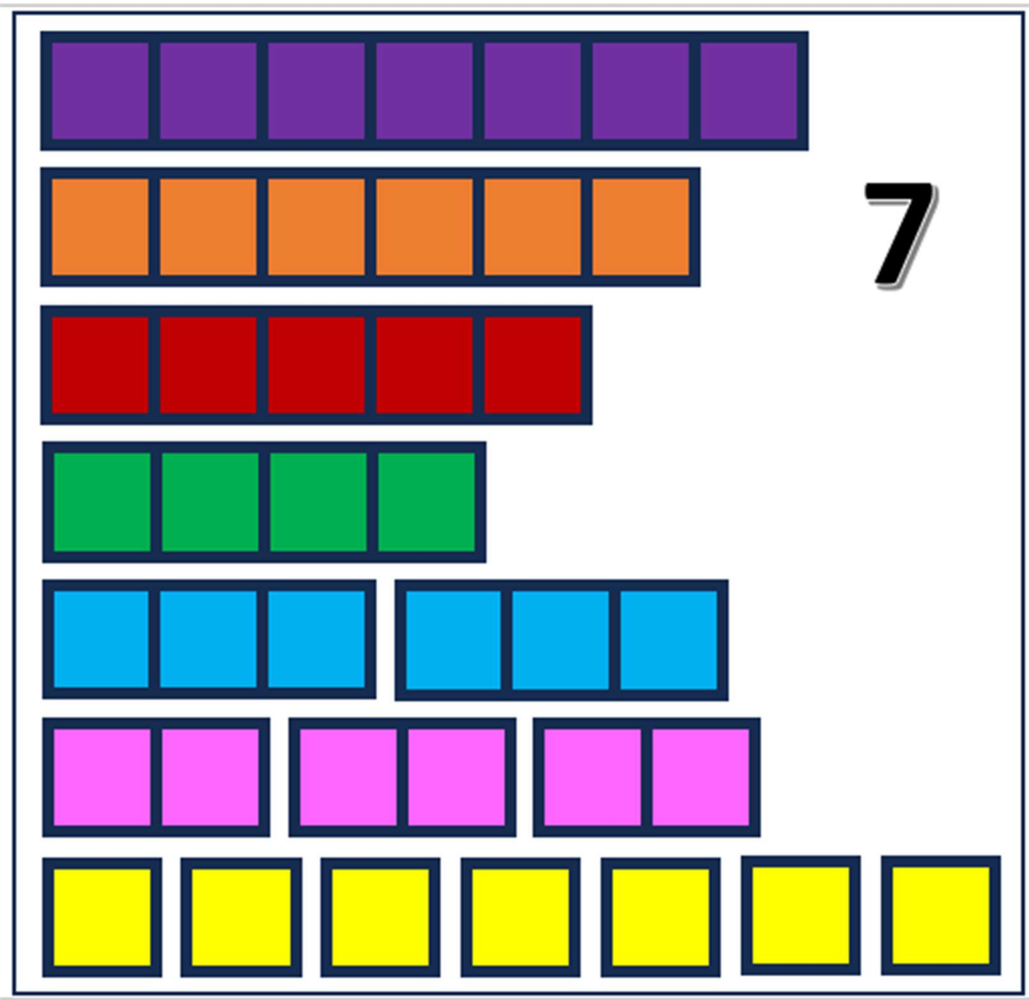
Appendix

Here are rectangles that you can cut out and use in counting partitions and compositions.





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Notes for Parents

Partitioning numbers is a very rich area. I wish that I had thought of this in my schooling but only saw this relatively recently when I attended with my then 6th or 7th grade daughter a presentation from Art of Problem Solving (where I also teach). I thought that this would be a fun area to explore with the students. Sure enough I've learned a lot while putting these materials together.

In the last class I shared a bit about the background of partition theory including references. I mentioned the partition function p(n) is the number of possible partitions without regard to order. The pattern is difficult to find but Leonhard Euler found it. Last time I mentioned the Hardy-Ramanujan approximation as well. As far as when order matters, your children will have discovered that the pattern is easy to find; the number of compositions, let's call it c(n), keeps doubling starting with c(1)=1. So for any positive integer n, c(n) = 2^{n-1}.

Leonhard Euler is one of the giants of mathematics; among his many accomplishments he is the inventor of graph theory, a. He found that if you look at the partitions of a positive integer n and count the ones that use only odd integers (e.g., 4 = 3+1 but not 2+1+1), the count of these odd partitions is the same as the count of the number of partitions where each number is distinct (e.g., 4 = 3+1 but not 2+1). This seems quite surprising to me. His result was generalized in 1883 by James Whitbread Lee Glaisher. If you are interested in seeing how he generalized this result, you can get a quick overview on Wikipedia[1].

Speaking of Euler, we might look at some graph theory, a field that he invented, this or in a future semester. It's a favorite topic of mine and I've introduced it to children as young as kindergarteners.

Computing p(n), the number of partitions of n

This is a BCMATH version of the BC program partition, which in turn is based on a BASIC program, which depends on Euler's recurrence relation

$$\sum_{i=0}^{n} \alpha(i)p(n-i) = 0, n \geq 1,$$

$$\text{where } p(0) = 1 \text{ and } \alpha(n) = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^r, & \text{if } n = \frac{r(3r\pm1)}{2}, r \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For example:

p(1)-p(0)	= 0, p(1) = 1
p(2)-p(1)-p(0)	= 0, p(2) = 2
p(3)-p(2)-p(1)	= 0, p(3) = 3
p(4)-p(3)-p(2)	= 0, p(4) = 5
p(5)-p(4)-p(3)+p(0)	= 0, p(5) = 7
p(6)-p(5)-p(4)+p(1)	= 0, p(6) = 11
p(7)-p(6)-p(5)+p(2)+p(0)	= 0, p(7) = 15
p(8)-p(7)-p(6)+p(3)+p(1)	= 0, p(8) = 22
p(9)-p(8)-p(7)+p(4)+p(2)	= 0, p(9) = 30
p(10)-p(9)-p(8)+p(5)+p(3)	= 0, p(10) = 42

See Hardy and Wright *An introduction to the theory of numbers* (1962 edition), p. 286 or Harald Scheid, *Zahlentheorie*, 2. Auflage, s. 387.
Also see a lecture by Ken Ono.

Enter n (1 ≤ n ≤ 10000): 12

[1] wikipedia.org/wiki/Glaisher%27s_theorem, accessed Jan. 29, 2024

In fact I gave a talk at a mathematics teachers conference where I described how I introduce graph theory in early elementary school and how that helps students in their problem-solving skills and in later mathematics. I'm happy to share a copy of the talk if you are interested and ask.

Getting back to the pattern for partitions, the graphic on the previous page is from a site[2] that illustrates a recurrence from Euler. This site also allows one to provide a value of n for it to calculate its number of partitions.

Euler derived a recurrence formula to find p(n). The excerpt copied to the right from a nice overview article *An Introduction to Integer Partitions* by Akintunde Ayodele from July 10, 2023[3] shows how you can apply this. Did I expect your children to come up with this? No! But I enjoy seeing how they try to find patterns. I might later this introduce pentagonal numbers. Later in their mathematical journey, your child may enjoy revisiting this topic.

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots$$

The numbers in the brackets are once again the pentagonal numbers, of form $k(3k-1)/2$, and the sign before each term is the sign of -1 rasised to the power of $|k| + 1$, where $|k|$ is the absolute value of k. With that you can continue the recurrence formula for as long as needed.

Notice how the tree-and-forest situation shows up again, this time in reverse - to get a tree, give the forest(or at least a part of it).

To calculate p(9) for example, we'll need to know p(8), p(7), etc. If those values are not already pre-computed, we may just start with p(0) = 1 and work upwards.

p(9) = p(8) + p(7) - p(4) - p(2) + p(-3) + p(-6) - ...

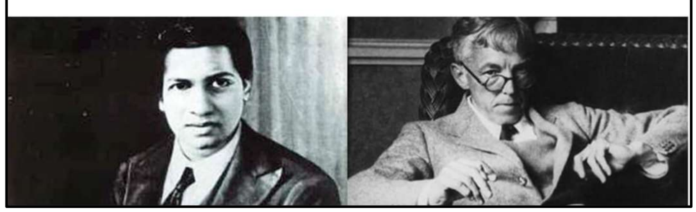
p(n) = 0 for negative values of n.

So, we are left with p(9) = p(8) + p(7) - p(4) - p(2).

By the way, something else that I wanted to bring to your attention about this "difficult" way of finding a formula for p(n) is that this is a great place where generating functions shine. If you've seen generating functions, you may appreciate a textbook-like treatment[4] and an article[5] showing how they apply to this problem. The Ayodele reference above is an easier introduction to this approach. Estimating p(n) is what the Hardy-Ramanujan Asymptotic Partition Formula does[6].

An asymptotic expression for $p(n)$ is given by

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \text{ as } n \to \infty$$



[2] numbertheory.org/php/partition.html, accessed Jan. 29, 2024
[3] medium.com/intuition/an-introduction-to-integer-partitions-799f934dd955, accessed Jan. 30, 2024
[4] whitman.edu/mathematics/cgt_online/book/section03.03.html, accessed Jan. 30, 2024
[5] math.berkeley.edu/~mhaiman/math172-spring10/partitions.pdf, accessed Jan. 30, 2024
[6] This picture is from twitter.com/SrinivasR1729/status/1254348168357048320/photo/1, accessed Jan. 30, 2024; In the last class notes I also referenced theoremoftheday.org/NumberTheory/Partitions/TotDPartitions.pdf, accessed Jan. 16, 2024

By the way, the Berkeley Math Circle has a handout[7] from Joshua Zucker on compositions and partitions that you may enjoy. There is also a nice comprehensive presentation[8] on partitions in general that the Berkeley Math Circle has that you might like to review.

There is a nice overview article, *The Symmetry of Young Diagrams*[9] that you may enjoy (you have to create a userid to read the full article); I may introduce this graphic way of depicting partitions. Another very nice, short article *Number Partitions: Euler's Astonishing Insight* that ties this all together is on the thatmath.com site[10].

Till next time!

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[7] mathcircle.berkeley.edu/sites/default/files/archivedocs/2011_2012/lectures/1112lecturespdf/BMC_Beg_Feb7_2012_FancierCountingI.pdf, accessed Jan. 30, 2024

[8] mathcircle.berkeley.edu/sites/default/files/handouts/2020/BMCUpper2020Fall_Partitions_compressed.pdf, accessed Jan. 30, 2024

[9] cantorsparadise.com/the-symmetry-of-young-diagrams-cee8b503d1c8, accessed Jan. 30, 2024

[10] thatsmaths.com/2021/12/23/number-partitions-eulers-astonishing-insight, accessed Jan. 30, 2024