CHMC Advanced Group:
All About Paradoxes

Feb 3, 2024

1 Introduction

In our last session we came across an example about disease testing with a seemingly counter-intuitive result in conditional probability. A test that was very efficient at determining whether or not someone had a disease seemed to be very poor at determining if someone testing positive actually had the disease. Of course, there were other factors at play (such as a low infection rate of the disease) that affected this outcome, but sometimes mathematical anomalies like this may have us questioning whether our work was correct. Today, we'll examine a few more mathematical paradoxes with applications to statistics, traffic, sports, and even physics.

2 Simpson's Paradox

Simpson's Paradox is a well-known phenomenon in statistics that can cause us to misinterpret a correlation between points in a data set. In particular, it can make data seem to have an *opposite* correlation when including or ignoring particular 'variables'. Let us look at some examples.

Exercise 2.1. You have just been promoted as the captain of an elite space vessel, and are tasked with implementing a new defense system for your ship. Before deciding which defense system to use, you run a few tests. In 1000 tests of the systems, the Sonic Blaster was able to stop the oncoming hazards in 790 of the tests, and the Laser Mesh was successful in 900 of the 1000 tests. What percentage of the tests were successful for the Sonic Blaster and for the Laser Mesh? Which defense system would you recommend for your crew?

Exercise 2.2. After closer examination, you notice that the tests could be broken down into stopping slow-moving asteroids, and fast-paced space debris. When you break down your tests into these two categories, the following data emerges:

Tests against Asteroids

Defense System	Successful Tests	Failed Tests	Total	Success Rate
Sonic Blaster	580	10	590	
Laser Mesh	860	30	890	

Tests against Space Debris

Defense System	Successful Tests	Failed Tests	Total	Success Rate
Sonic Blaster	210	200	410	
Laser Mesh	40	70	110	

Fill in the table with the success rates of each system for each test.

Exercise 2.3. Do you notice anything strange with the percentages in these tables compared to the percentages you found in Exercise 2.1?

Exercise 2.4. Does this change your mind about which system you would recommend for your crew?

Exercise 2.5. A UNC basketball player and a Duke basketball player have a contest to see who can make the most free throws out of 1500 shots taken over the course of 2 days. During the first day, the UNC player takes 1200 shots and she makes about 62.2% of them, and the Duke player makes about 63.6% of the 700 shots she takes. During the second day, the UNC player takes 300 shots and she makes about 58.3% of them while the Duke player takes 800 shots and makes about 58.8% of them. Who wins the contest? Which player is the better free throw shooter?

Exercise 2.6. Suppose you have 4 boxes, labeled A, B, C, D, each with some number of red and blue marbles according to the following chart:

	Box A	Box B	Box C	Box D
Red Marbles	70	y	2	7
Blue Marbles	30	3	98	53

Suppose the following criteria are true:

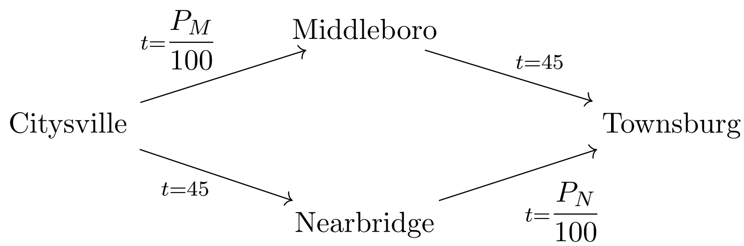
- The probability of selecting a red marble from box A is *less than* the probability of selecting a red marble from box B.
- If we combine boxes A and C into one box AC and combine boxes B and D into one box BD, then the probability of selecting a red marble from box AC is *greater than* the probability of selecting a red marble from box BD.

What are all possible values of y (the number of red marbles in box B) so that both of these criteria are satisfied?

3 Braess's Paradox

Braess's Paradox can be stated simply by saying "increasing the capacity of a network may actually decrease its efficiency." Have you ever been stuck in traffic and thought to yourself "if only they would only open up more roads; then I could get home faster"? We'll observe an example where that may not always be the case.

Consider the road network in the diagram below in which each day 4000 drivers must make their way from Citysville to Townsburg. Let P_M represent the number of people traveling on the road from Citysville to Middleboro and P_N represent the number of people traveling on the road from Nearbridge to Townsburg. Then, the time it takes to travel from Citysville to Middleboro is $\frac{P_M}{100}$ minutes, and the time it takes to travel from Nearbridge to Townsburg is $\frac{P_N}{100}$ minutes. The roads from Citysville to Nearbridge and from Middleboro to Townsburg both take a constant 45 minutes.



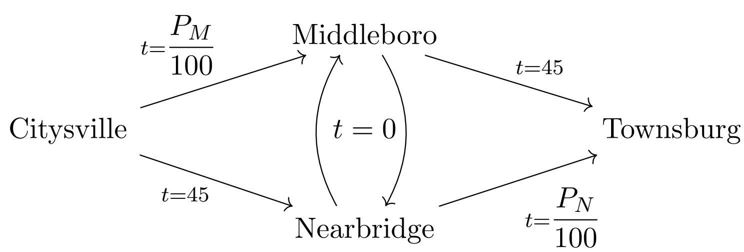
Exercise 3.1. As we can see above, the time it takes each commuter to travel from Citysville to Townsburg depends both on which route they choose *and* how many people choose the same route as them. Fill out the following table of commute times depending on the number of people who choose each path:

P_M	P_N	Time to travel $C \to M \to T$	Time to travel $C \to N \to T$
0	4000		
200	3800		
500	3500		
1000	3000		
1500	2500		
1800	2200		
2000	2000		
2200	1800		
2500	1500		
3000	1000		
3500	500		
3800	200		
4000	0		

Exercise 3.2. After each day, each commuter will look at their own commute time as well as the time it would have taken them if they had gone the other route. If they chose the faster way, then they will go the same way the next day. If they chose the slower way, then they *may* choose the other direction the next day, or they may choose the same way two days in a row.

We say that this system reaches a **Nash equilibrium** if nobody would be better off by switching their choice, at which point they will make this same decision every day. In the Nash equilibrium for this system, how many people will choose to take the route from Citysville via Middleboro? How long will it take them to make the trip from Citysville to Townsburg?

Now, the department of transportation has decided to try to make traffic flow a little more smoothly, and has built an incredible perfect road between Middleboro and Nearbridge. Commuters can now travel between these two cities in either direction in literally no time at all.



This allows commuters to choose their route in two distinct choices. First, they can choose to go from Citysville to Middlesboro or from Citysville to Nearbridge. Then, (depending on whether they take the fancy new road) they can choose to go from Middlesboro to Townsburg or from Nearbridge to Townsburg.

Exercise 3.3. Fill in the following table of travel times via the first leg of the trip (i.e. the non-stop portion of the trip that will take them either from Citysville through Middleboro or from Citysville through Nearbridge).

P_M	Time to travel $C \to M$ directly	Time to travel $C \to N$ directly
0		
500		
1000		
1500		
2000		
2500		
3000		
3500		
4000		

Exercise 3.4. What is the Nash equilibrium for the first leg of the trip (i.e. the point where everybody is happy with their choice of route to one of the two intermediary cities)? How many people will choose to travel on the direct road from Citysville to Middleboro? How long will this take?

Exercise 3.5. Fill in the following table of travel times for the second leg of the trip

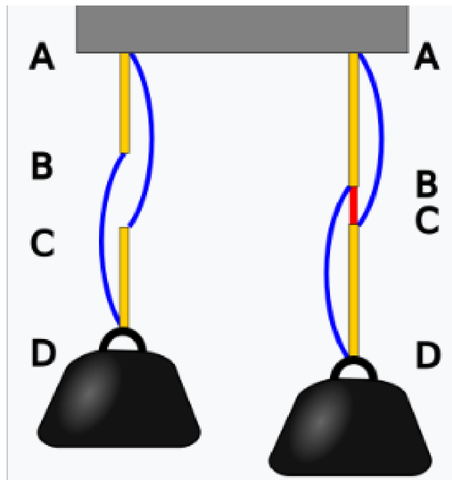
P_N	Time to travel $M \to T$ directly	Time to travel $N \to T$ directly
0		
500		
1000		
1500		
2000		
2500		
3000		
3500		
4000		

Exercise 3.6. What is the Nash equilibrium for the second leg of the trip (i.e. the point where everybody is happy with the road they chose from one of the intermediary cities to Townsburg)? How many people will choose to travel from Middlesboro to Townsville? How many people will take the fancy new road and then the road from Nearbridge to Townsville? How long will this take?

Exercise 3.7. Based on your answers to exercises 3.4 and 3.6, what will everyone's commute time be? Is this better or worse than without the new road?

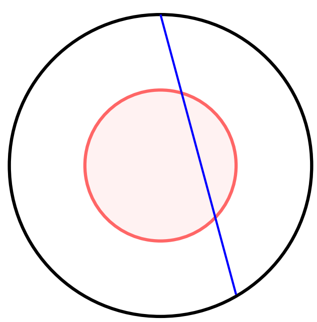
The basis of this paradox is that drivers act very selfishly. If everyone would agree never to use the new road, then everyone's commute times would collectively be lower, but the presence of the road incentivizes drivers to use it in order to shorten their own commute, to the detriment of everyone else.

This paradox not only plays out with people, but can occur in physical demonstrations as well. Consider the following system: A weight is hanging from a platform, suspended by two springs (connecting points A to B and points C to D), two loose strings (connecting points A to C and points B to D), and one taut rope connecting points B and C. The presence of this taut rope, however, prevents the loose strings from being used to their full capacity, so cutting the rope actually causes the weight to *rise* rather than fall.



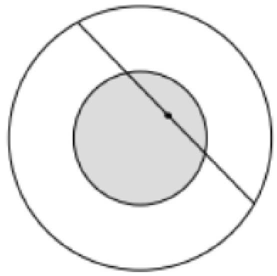
4 Bertrand's Paradox

In the late 1800's, mathematician Joseph Bertrand presented the following problem (or at least one equivalent to this one). Suppose you have a circle of radius 1 inside of a circle of radius 2 so that the two circles are centered on the same point. We then, at random, draw a chord of the larger circle. What is the probability that the chord touches the inner circle?

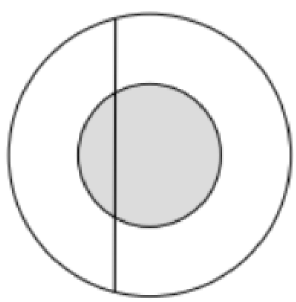


We will look at three different approaches to answering the question:

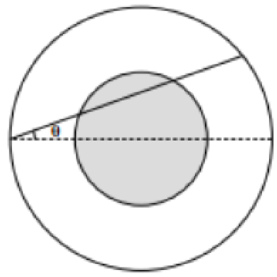
Exercise 4.1. Any chord of a circle is uniquely determined by its midpoint (unless the midpoint happens to be the center of the circle, making the chord a diameter of the circle, but let us ignore this for now). Therefore, any chord drawn that touches the inner circle must have its midpoint inside the inner circle. Given this method of choosing a random chord by choosing a point in the outer circle at random, what is the probability that the midpoint of the chord lies in the inner circle?



Exercise 4.2. The circles can always be rotated so that any chord is a vertical line segment. In this way, any chord of a circle is uniquely determined by the horizontal position of this vertical line. Given this method of choosing a random chord by randomly choosing an x-coordinate of this vertical line segment, what is the probability that the chord touches the inner circle?



Exercise 4.3. Define a horizontal line that goes through the center of the circles. The circles can always be rotated so that one point on the chord lies on this horizontal line on the left side of the circle. You can then measure the angle θ created between the chord and the horizontal line. The possible values of θ lie between $-90°$ and $90°$. What possible values of θ result in the chord touching the inner circle? What is the probability that the chord touches the inner circle?



This may feel rather unsatisfying and/or confusing, but it turns out that the ambiguity lies in what we mean when we use the phrase "at random". If we give a precise meaning to the random selection process, then this problem will have a well-defined solution, but until then, it could be very ambiguous.

Exercise 4.4. In Exercise 4.1, we claimed that any chord of a circle is uniquely determined by its midpoint, so long as that midpoint is not the center of the circle. Can you prove this fact? That is, given any point P inside a circle which is not the center, how would you construct a chord of the circle such that P is its midpoint? Try drawing a picture to see what you can come up with.