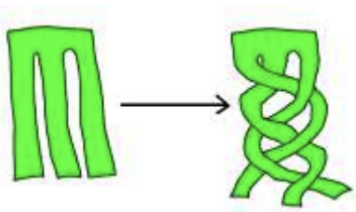
Braid Puzzles[1]

1 Possible or Impossible Braids

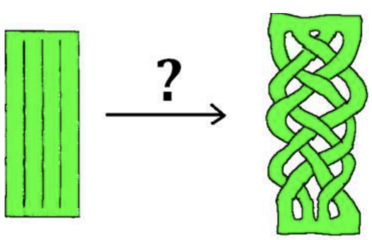
1. Make a braid by starting with three parallel strands joined together at one end but kept loose at the other:



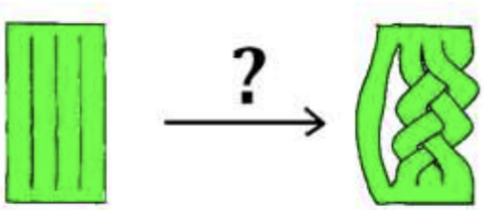
2. But why bother with the loose ends? Go ahead and make a braid with no free ends! It can be done!

Notice that the individual strands are relatively flat: there are no twists and the same one side of the paper faces outwards at all times within each strand.

3. Why stop at three strands? Can you make this no-free-end version of a four-strand braid? Try it!

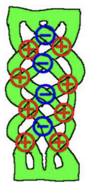


4. What about this one?



[1] copied from James Tanton: Solve This! and Math Galore!

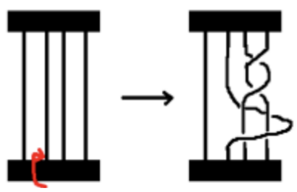
5. Let's look at the four-strand braid. Notice that strands cross a total of 12 times, sometimes with the right strand crossing over the left (let's call these positive crossings) and sometimes with the left strand over the right (negative crossings).



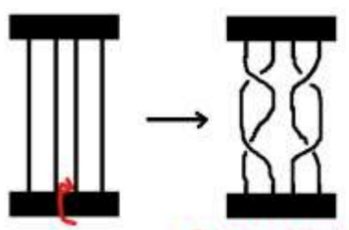
Index = +4

6. Here are some possible moves with four strands. What is the crossing number of each one?

MOVE 1: *Push the bottom end of the felt through a slit to the left or to the right.*

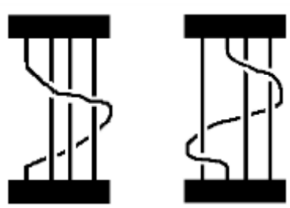


MOVE 2: *Push the bottom end of the felt through the middle slit.*

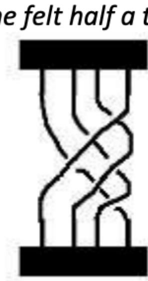


MOVE 3: *Pick up and move one strand around the base of the object.*

Here are two typical examples of this move.



MOVE 4: *Rotate the bottom half of the felt half a turn.*



7. *Can you find more moves or prove that these are the only ones?

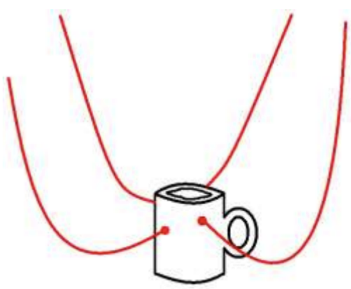
8. If we assume that these are the only types of moves, how can we use that to prove that the four braid (with all strands braided) is impossible?

9. *Demonstrate the 4 strand braid with only 3 strands actually braided is possible, or else prove that it is impossible.

10. *What about 5 strand braids?

2 Teacup Twists

11. Hold a teacup up in the air in the center of a room and have friends tape four or five strings from the cup to various points about the room. (Just two or three strings make too easy a puzzle.) Be sure to leave plenty of slack in the strings.



Rotate the teacup one full turn, 360°, tangling the strings in the process.

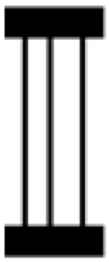
From this point on, the cup is to be held fixed in space, never to move again!

Your job is to now maneuver the strings around the cup and untangle them. (The person holding the cup will have to move their hands out of the way as folk bring strings up and over or down and around the cup. But the cup itself is not to move from its position in space, nor turn or tilt in any way.) Can you and the team untangle the strings?

12. Now try the following: Give the cup another full turn IN THE SAME DIRECTION, tangling the strings even further! It should now be twisted 702° from the original. Again holding the cup fixed in space, maneuver the strings around the cup and untangle them. This second task can definitely be done! Try it!

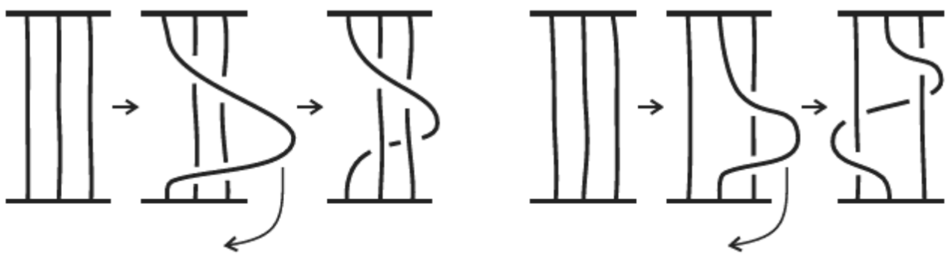
13. Explain why if it is impossible to untangle just three strings taped to a teacup under one full turn, it is thus impossible to tangle more than three strings taped to a teacup under one full turn.

Here is a schematic of three strings tied to a teacup.



(a) . Following the work of the previous section, what is the index of tangle produced by one full turn?

(b) Following the previous section ... Only moves of type 3 are permitted in trying to untangle the strings of a teacup. What is the index of any such move?



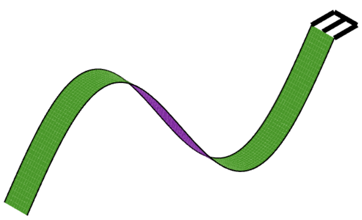
There are also moves that can be performed within a braid:

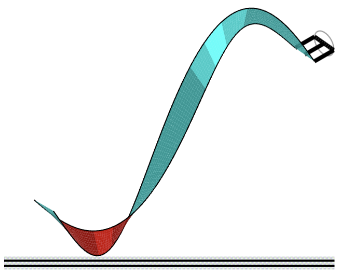
(c) Explain why it is impossible to untangle three strings tied to a teacup that have undergone one full turn.

14. Is it possible to untangle when you do two full turns, with any number of strings? Explain.

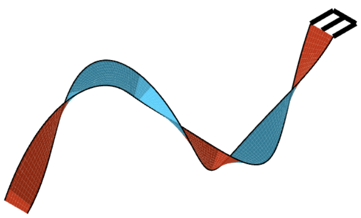
3 More Puzzles

15. Take a belt and stretch it out, with one person holding each end. Twist it by 360°. Now, holding the ends flat (no more twisting), remove the kink. Can it be done?





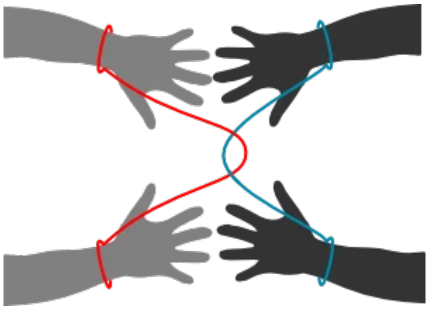
16. What if you twist the belt by 720°? Now can you undo the kink without any more twisting?



17. What does this have to do with untwisting a teacup?

18. Wrap a rubber band around the end of a pencil so that the band always lies flat agains the wood. How many times *must* the band wrap around the pencil to achieve this?

19. Tie a string around two people's wrists. Then tie another string around another person's wrists, so that the strings link. Can you separate yourselves without cutting the string?



20. Pick up a piece of string from a table top, one end of the string with your left hand and the other end with your right hand. Now, without ever letting go of either end of the string, maneuver your arms and your body so as to eventually tie a knot in the string. It can be done!

4 More Braids

21. Take three strings, two colored red and one yellow, and tie them to the back of a chair so that the yellow strand lies in the middle position. Braid the three strands in any manner you care to choose. That is, cross adjacent strands over or under each other in any organized or disorganized fashion. Make sure when you are done that the yellow strand is in the middle position. Tie the three ends to a pencil.

    Can you untangle any such braid by maneuvering the pencil back and forth between the strands?

22. Here is a way to analyze braids on 3 strands. We can read the braid from top down as a sequence of crossings. Let L denote the crossing of a strand in the left-most position over or under the strand in the middle position and R the corssing of the right two strands.

23. Check that you can always convert an L undercrossing to an L overcrossing and vice versa (and same for a right undercrossing and overcrossing) by moving the pencil. Therefore it doesn't matter which one is which if we are trying to write down which braids can be untangled.

24. Check that two consecutive L crossings can be undone by moving the pencil.

25. Check that a sequence LRL can be undone by moving the pencil.

26. Now which braids (sequences of L's and R's) can be undone by moving the pencil?

Hints: