CHMC Advanced Group:
Generating Functions

Nov. 4, 2023

1 Introduction

Last time, we took a look at the idea of generating functions−power series of the form $\sum_{n=1}^{\infty} a_n x^n$ that we can use to encode information about certain sequences of numbers. Before, we looked at them in the context of partitions and other counting problems. Today, we will look at generating functions where (a_n) represents a sequence defined by some kind of recursive relationship.

2 Fibonacci Numbers

We will consider the famous sequence known as the Fibonacci numbers which is defined by a **recursion** (i.e. values in the sequence can be defined based on values earlier in the sequence). The equation is as follows:

$$a_0 = 0, a_1 = 1, \text{ and } a_{n+1} = a_n + a_{n-1} \text{ for } n \geq 1.$$

Exercise 2.1. Determine $a_0, a_1, a_2, \ldots, a_{12}$ using the formula above.

Exercise 2.2. Suppose we have a generating function $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Write out the first few terms of this series. (Note: for the Fibonacci sequence, note that we're starting this index at $n = 1$ because the $n = 0$ term is $a_0 = 0$)

Exercise 2.3. For each $n \geq 1$, multiply the terms in $a_{n+1} = a_n + a_{n-1}$ by x^n, and add all of these equations together to get

$$\sum_{n=1}^{\infty} a_{n+1}x^n = \sum_{n=1}^{\infty} a_n x^n + \sum_{n=1}^{\infty} a_{n-1}x^n.$$

Now, rewrite each side of this equation in terms of x and $f(x)$. (Hint: the left side should look like a rational function)

Exercise 2.4. With the equation you found in Exercise 2.3, solve for $f(x)$ in terms of x. This should look like a rational function where the numerator is a linear factor and the denominator is a quadratic polynomial.

Exercise 2.5. Find the roots of the quadratic polynomial in the denominator of your expression found in Exercise 2.4. Use this to rewrite the denominator as $(1 - r_1 x)(1 - r_2 x)$ where r_1, r_2 are the roots.

Exercise 2.6. Show that if $x^2 + bx + 1 = (1 - sx)(1 - tx)$, then

$$\frac{x}{x^2 + bx + 1} = \frac{1}{s - t}\left(\frac{1}{1 - sx} - \frac{1}{1 - tx}\right)$$

Exercise 2.7. Show that for the infinite series defined by $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$, this can be written as $\frac{1}{1-\frac{1}{2}} = 2$. To do this, suppose $S\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$. What is $\frac{1}{2} \cdot S\left(\frac{1}{2}\right)$? Now, compute $S\left(\frac{1}{2}\right)(1 - \frac{1}{2}) = S\left(\frac{1}{2}\right) - \frac{1}{2} \cdot S\left(\frac{1}{2}\right)$. From this, solve for $S\left(\frac{1}{2}\right)$ in terms of a real number.

We'll repeat the same exact process, but this time for an arbitrary real number r (just like last week, we'll forget about convergence and just consider this as a formal power series; this technically only converges when $|r| < 1$, but let's not worry about that for now).

Exercise 2.8. Show that for the infinite series defined by $\sum_{n=0}^{\infty} r^n$, this can be rewritten as $\frac{1}{1-r}$. To do this, suppose $S(r) = \sum_{n=0}^{\infty} r^n$. What is $rS(r)$? Now, compute $S(r)(1-r) = S(r) - rS(r)$. From this, solve for $S(r)$ in terms of r.

Exercise 2.9. Apply the formula from the previous exercise to the terms $\frac{1}{1-r_1 x}$ and $\frac{1}{1-r_2 x}$ to get series of the form $\sum_{n=0}^{\infty} b_n x^n$ and $\sum_{n=0}^{\infty} c_n x^n$ respectively. Now apply Exercises 2.5 and 2.6 to simplify the expression from Exercise 2.4. Combine all of these series to get an equation for $f(x) = \sum_{n=0}^{\infty} a_n x^n$. This determines a formula for a_n for all $n \geq 0$ that doesn't require recursion.

3 Catalan Numbers

Consider for a positive integer n a group of $2n$ points spaced equally around a circle. For such an arrangement, consider connecting any pair of points by drawing a line between them inside of the circle. Keep doing this until all of the points have been paired with another. We say a diagram is valid if every point is paired with another, no point is paired with one other point, and none of the lines drawn cross each other.

Consider, for example $n = 2$ and points placed at $3, 6, 9$, and 12 on a clock. Valid arrangements are $(3, 6)(9, 12)$ and $(3, 12)(6, 9)$ since every point is paired with exactly one other point and none of the lines cross. An invalid arrangement is $(3, 9)(6, 12)$ because the lines cross.

We want to determine c_n, the number of valid arrangements of $2n$ points on the circle. By convention, $c_0 = 1$ and $\binom{0}{0} = 1$.

Exercise 3.1. Calculate c_0, c_1, c_2, c_3, and c_4. It is highly encouraged to draw out these diagrams to keep track of all the possible valid arrangements.

Now suppose I know the values of $c_0, c_1, \ldots, c_n$ and I want to determine the value of c_{n+1}. We will attempt to use a recursion to relate the value c_{n+1} to the calues $c_0, c_1, \ldots, c_n$. Before attempting to do this abstractly, let's compute this by hand for small values of n.

Exercise 3.2. Try computing c_3 using the values of c_1 and c_2. Since lines connect a pair of points, consider which points of $2, 3, 4, 5$, or 6 can be connected to point 1 that will make a valid arrangement. Consider the points to the left and to the right of the line connecting point 1 to the other point. How many points are on the left and right side of this line? Now how many arrangements can be made on the left and right sides of the line? How do these relate to c_0, c_1, and c_2? Furthermore, choosing an arrangement on the left is independent of choosing an arrangement on the right, therefore we can multiply the number of arrangements on the left by the number of arrangements on the right to determine the total number of arrangements for a particular choice of point to connect to point 1. We can then add up all of these products to compute c_3.

Exercise 3.3. Count by hand and use the process above to compute c_4. Check that they match.

Exercise 3.4. Determine for a general positive integer n how to compute the value of c_{n+1} based on the values $c_0, c_1, \ldots, c_n$.

(Note: The algebra on the generating functions is about to get a little heavy, so if you'd rather skip ahead and do Exercises 3.9 and 3.10 in section 3.1 before coming back here, that may be more fun)

Exercise 3.5. Suppose the generating function of the Catalan numbers is given by $C(x) = \sum_{n=0}^{\infty} c_n x^n$. Consider $C(x)^2$ and group the coefficients according to the power of x (you should only need to do this for low powers of x to see the pattern). What pattern do you see? Is there a power of x you can multiply $C(x)^2$ so that it looks similar to $C(x)$? If so, what is it? How can we use this to recover $C(x)$?

Exercise 3.6. Use the quadratic equation in the previous example to solve for $C(x)$. You will get two solutions (by nature of the quadratic formula) but for reasons beyond the scope of this worksheet, we'll take the solution where the square root is subtracted.

To help us determine the coefficients of this series in a non-recursive way, we use the identity

$$\sqrt{1+y} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^n(2n-1)} \binom{2n}{n} y^n$$

Exercise 3.7. In the above identity, replace $y = -4x$. Then multiply by -1 and add 1. Finally, divide by $2x$. Next, reindex so that the index starts at $n = 0$.

Exercise 3.8. Show that

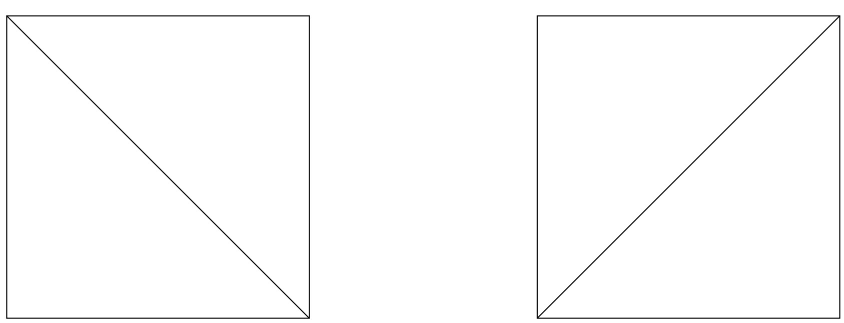
$$\frac{1}{2}\frac{1}{2n+1}\binom{2n+2}{n+1} = \frac{1}{n+1}\binom{2n}{n}.$$

Using this and the series you computed in the previous exercise, what is the value for c_n for $n \geq 0$?

3.1 Other Formulations of the Catalan Numbers

The Catalan numbers not only count the number of valid arrangements of lines connecting pairs of points around a circle, but in fact they count over 200 different combinatorial objects that mathematicians are interested in. We will look at two more of them.

Consider a regular $(n+2)$-sided polygon. We aim to **triangulate** this polygon, that is, draw $n-1$ diagonals inside of the shape so that none of the diagonals cross and the polygon is divided into n triangles. For example, for $n = 2$, there are $c_2 = 2$ ways to triangulate a square:



The Catalan number c_n counts the number of ways to triangulate this $(n+2)$-sided polygon.

Exercise 3.9. Using the values you've computed, draw the c_3 number of ways to triangulate a regular pentagon and the c_4 number of ways to triangulate a regular hexagon.

Now consider the product of $n+1$ numbers $x_1 \cdot x_2 \cdot \cdots \cdot x_{n+1}$. Even though multiplication is associative, we can draw $n-1$ pairs of parentheses to specify the order in which we would like to multiply them (or n pairs if we draw another pair of parentheses around the whole product). For example, there are $c_2 = 2$ ways to specify an order to multiply three numbers a, b, c:

$$a(bc) \qquad (ab)c$$

In general, there are c_n ways to draw these pairs of parentheses to specify an order to multiply these $n+1$ numbers.

Exercise 3.10. Using the values you've computed, draw the c_3 ways you can draw parentheses around the product $a \cdot b \cdot c \cdot d$ and the c_4 ways you can draw parentheses around the product $a \cdot b \cdot c \cdot d \cdot e$.

Exercise 3.11. Even though we know that both of these are counted by the Catalan number c_{n-1}, prove that the number of ways to triangulate the $(n+1)$-gon is the same as the number of ways to draw parentheses around the product $x_1 \cdot x_2 \cdot \cdots \cdot x_n$. That is, describe a process that given a particular triangulation of a polygon, I can associate that to a particular arrangement of parentheses of a product of numbers and vice-versa. Can you describe a process to show that either of these these is also the same as the number of arrangements of connecting $2n-2$ points around a circle that we described at the beginning of Section 3?