CHMC Advanced Group:
Generating Functions

Oct. 21, 2023

1 Introduction

In mathematics certain sequences of numbers are considered interesting because of the properties they possess (e.g. the sequence of prime numbers, the sequence of triangular numbers, the Fibonacci numbers). It is often useful to encode the information of a sequence into a formal power series, that is, for a sequence $\{a_k\}_{k=0}^{\infty}$ there is an associated power series $\sum_{k=0}^{\infty} a_k x^k$. As an example, my favorite sequence - the Catalan numbers - begins $1, 1, 2, 5, 14, 42, \ldots$, and its associated power series is $1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + \cdots$. Now we can study the sequence by studying this power series, and one approach is to use a combinatorial tool known as a **generating function**. Today, we will examine the use of generating functions to tackle some (possibly familiar) problems.

Warm-up Questions

Exercise 1.1. How many different ways are there of making change for \$1.00 using only nickels, dimes, and quarters? What about \$2.00?

Exercise 1.2. You are ordering pizzas for your math club meeting, but some of your friends are *a little particular*. Your local pizza place offers three possible toppings: sausage, pineapple, and mushrooms. In order to satisfy the math club, you must order according to these rules:

- You will order 10 *single-topping* pizzas (every pizza will only have one topping).
- You have some vegetarians in the group, so at least half the pizzas must be vegetarian (i.e. no more than 5 will be sausage pizzas).
- You must order an **even** number of mushroom pizzas
- You must order a **prime** number of pineapple pizzas (for our purposes, 0 and 1 are **not** prime)

How many different possible ways could you make this order and satisfy all of these rules?

2 Partitions

Let n be a positive integer and consider a pile of n pencils. We may wish to separate these pencils into non-empty groups of pencils in such a way that the first group has at least as many pencils as the second group, the second has at least as many as the third, and so on. The sizes of these groups then give what is known as a **partition** of the integer n. For example, when $n = 5$, some possible partitions may be $3, 2$, or $4, 1$ or $2, 1, 1, 1$. Note that we consider partitions to be *unordered*−that is, $2, 1, 1, 1$ and $1, 2, 1, 1$ represent the *same* partition of 5. An interesting question one could ask (and was asked often historically) is "given an integer n, how many partitions of n are there?"

Exercise 2.1. Determine the number of partitions for $n = 1, n = 2$, and $n = 3$. It may help to develop a systematic approach (perhaps either by the size of the largest group or by the number of groups).

Exercise 2.2. Determine the number of partitions for $n = 4$, $n = 5$, and $n = 6$. Here it will really help to use a systematic approach to keep track of everything.

Now consider the monomial x^n and the possible ways to multiply monomials of the form x^{k_i} such that $x^{k_1} \cdot x^{k_2} \cdot \cdots \cdot x^{k_j} = x^n$ and $k_1 \geq k_2 \geq \cdots \geq k_j$.

Exercise 2.3. How does finding the number of possible ways to multiply these monomials relate to our earlier problem about the number of partitions? What happens as this j varies from $j = 1$ to $j = n$?

We can connect these two ideas using something called a **generating function**. In particular, if we let a_n be the number of partitions of n, then we hope to write a function $f(x) = \sum_{n=1}^{\infty} a_n x^n$ in some meaningful way.

Remark. By this notation $\sum_{n=1}^{\infty} a_n x^n$, we mean the infinite sum $a_1 x + a_2 x^2 + a_3 x^3 + \cdots$. With series like this, we could ask for which values of x does this infinite sum equal a real number. In this case we say the series **converges** (otherwise, we say the sum **diverges**). But with formal power series like this, we kind of don't care. Instead of concerning ourselves with the convergence or divergence of this sum, we're more interested in the properties of the coefficients of the series and may use the properties of the sequence to determine other coefficients. In other words, it's usually convenient to think of these as some "polynomial of degree ∞".

Consider as before, writing the partitions of a positive integer in terms of the size of the largest group. Say for the partition of $n = 7$, we have the partition $3, 2, 2$. In this partition, we first have a group of 3, then a group of 2, then another group of 2. We can write this equivalently as $x^3 \cdot x^2 \cdot x^2 = x^7$. Convince yourself that this correspondence between partitions and products is unique in the sense that to each partition there is a corresponding product and to each product there is a corresponding partition.

Exercise 2.4. Fix $n = 4$. Consider the product of $(1+x+x^2+x^3+x^4)(1+x^2+x^4)(1+x^3)(1+x^4)$. What is the coefficient on x^4 when you expand this product? Compare this to the number a_4 you computed earlier. Try to relate the monomial products in this multiplication to the partitions. Now do the same for the x^3 coefficient in $(1 + x + x^2 + x^3)(1 + x^2)(1 + x^3)$. What do you observe?

Exercise 2.5. For $n = 5$ and $n = 6$, what product of polynomials and what specific terms do you suggest we check to find the number of partitions of $n = 5$ and $n = 6$. Now also check the coefficients on x, x^2, x^3, and x^4 in these products and determine how these are related to the a_n's you previously computed.

Exercise 2.6. How do you think the generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ can be represented as a product of infinite sums of monomials (possibly an infinite product)? Try to justify your answer.

Exercise 2.7. For positive integer k, it is known that $1 + x^k + x^{2k} + \cdots = \dfrac{1}{1 - x^k}$ (at least when you restrict to $-1 < x < 1$, but again, we don't really care about that right now). How can you write your answer to the previous exercise in more concise notation using this fact?

Now, let us consider partitions where we only allow odd-sized groups. For example, when $n = 3$ all such partitions are 3, and $1, 1, 1, 1$ and for $n = 4$, all such partitions are $3, 1$ and $1, 1, 1, 1$.

Exercise 2.8. Determine the number of partitions consisting only of odd-sized groups for $n = 2, 3, 4, 5, 6$.

Exercise 2.9. Consider the product $(1 + x + x^2 + x^3 + x^4 + x^5 + x^6)(1 + x^3 + x^6)(1 + x^5)$. How do the first few coefficients relate to the numbers of partitions of only odd-sized groups that you previously computed.

Exercise 2.10. For the generating function $f(x) = \sum_{n=0}^{\infty} o_n x^n$, (where o_n is the number of partitions of n consisting only of odd-sized groups), what product of infinite sums would you expect to result in this generating function?

Exercise 2.11. For the generating function $f(x) = \sum_{n=0}^{\infty} e_n x^n$, (where e_n is the number of partitions of n consisting only of even-sized groups), what product of infinite sums would you expect to result in this generating function?

Now, let's try to adapt our strategy for partitions whose groups are *distinct*. For example, for $n = 6$, such possible partitions are $5, 1$ and $4, 2$ and $3, 2, 1$.

Exercise 2.12. Determine the number of such partitions d_n for $n = 2, 3, 4, 5, 6, 7$. Then, relate these partitions to products of monomials as before. As with the last two examples, what infinite product of polynomials would we expect would give us the generating function $f(x) = \sum_{n=0}^{\infty} d_n x^n$?

2.1 Return to Warmups

Exercise 2.13. Use a generating function to help you solve Exercise 1.1. Write the generating function you would use and determine which coefficient you should look at. In order to do the actual computation, you can use Wolfram Alpha with a command that looks something like "expand *insert generating function here*" and click "more terms" under "series expansion at $x = 0$" until it gets to the coefficient you want.

Exercise 2.14. Use a generating function to help you solve Exercise 1.2. Write the generating function you would use and determine which coefficient you should look at. Again, you can offload all of the actual multiplication here to Wolfram Alpha.