Remainders and Divisibility, Part 2

October 21, 2023

1 Definitions

What is the remainder when we divide 18 by 5? [1]

What does it mean to divide a natural number N by a natural number m with remainder r?

For two numbers A and B, we say that $A \equiv B \pmod{m}$ if A and B have the same remainder when divided by m.

For example, $15 \equiv 3 \pmod{12}$.

- $8 \equiv 23 \pmod{5}$. Why?
- $8 \not\equiv 14 \pmod{5}$. Why not?

1. (a) Is $85 \equiv 0 \pmod{5}$?
   (b) Is $17 \equiv 73 \pmod{7}$?
   (c) Is $15 \equiv 2 \pmod{4}$?
2. (a) If two numbers are equivalent mod 11, what can we say about their difference?

   (b) If 11 divides $A - B$, will is necessarily be true that $A \equiv B \pmod{11}$?

3. Find positive numbers x and y such that
   (a) $-1 \equiv x \pmod{3}$
   (b) $-3 \equiv y \pmod{8}$
4. Without doing a lot of arithmetic, compute
   (a) $325 + 292 \pmod{3}$
   (b) $19 \cdot 17 \pmod{3}$

   Why does this shortcut method work?

   Lemma on Remainders: The sum of any two natural numbers has the same remainder, when divided by 3, as the sum or their remainders.

   The product of any two natural numbers has the same remainder, when divided by 3, as the product of their remainders.

   Proof:

[1] This week's problems are from *Mathematical Circles: The Russian Experience*

5. Find the remainder when
   (a) the number $2021 \cdot 2022 \cdot 2023 + 2024^3$ is divided by 7
   (b) the number 9^{100} is divided by 8

2 Case Work

6. Prove that the number $n^3 + 2n$ is divisible by 3 for any natural number n.
7. Prove that $n^5 + 4n$ is divisible by 5 for any integer n.
8. Prove that $n^2 + 1$ is NOT divisible by 3 for any integer n
9. If the natural numbers x, y, and z satisfy the equation $x^2 + y^2 = z^2$, prove that at least one of the numbers is divisible by 3.
10. Prove that $p^2 - 1$ is divisible by 24 if p is a prime number greater than 3.

3 Powers and Remainders

11. Find the last digit of the number 2023^{2023}
12. Find the last digit of the number 2^{50}
13. What is the last digit of 777^{777}?
14. Find the remainder when 2^{100} is divided by 3.
15. Find the remainder when 3^{2023} is divided by 7.
16. Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

4 Guess the Number

17. Given that p, $p + 10$, and $p + 14$ are prime numbers, find p. Hint: find remainders when divided by 3.
18. Given that p, $2p + 1$, and $4p + 1$ are prime numbers, find p.
19. If p, $4p^2 + 1$, and $6p^2 + 1$ are prime numbers, find p.
20. Find the smallest natural number which has a remainder of 1 when divided by 2, a remainder of 2 when divided by 3, a remainder of 3 when divided by 4, a remainder of 4 when divided by 5, and a remainder of 5 when divided by 6.

21. The prime numbers p and q and the natural number n satisfy the following equality:

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{pq} = \frac{1}{n}$$

Find the numbers.

22. Prove that there is a natural number n such that the numbers $n+1, n+2, n+3, \cdots n+2017$ are all composite.

5 Divisibility

23. Do there exist natural numbers a and b such that $ab(a-b) = 45045$?

24. Let us denote the sum of three consecutive natural numbers by a, and the sum of the next three consecutive natural numbers by b. Can the product ab be equal to 111111111?

25. Prove that the last non-zero digit of the number 1985! is even.

26. The natural numbers x and y satisfy the relation $34x = 43y$. Prove that the number $x + y$ is composite.

27. Prove that a natural number written using one 1, two 2's, three 3's, ..., nine 9's cannot be a perfect square.

28. Each of the natural numbers a, b, c, and d is divisible by $ab - cd$. Prove that $ab - cd$ equals either 1 or -1.

29. In a certain country, banknotes of four types are in circulation: 1 dollar, 10 dollar, 100 dollar, and 1000 dollar bills. Is it possibly to pay one million dollars using exactly half a million notes?

30. The number 1 is written on a blackboard. After each second, the number on the blackboard is increased by the sum of its digits. Is it possible that at some moment, the number 123456 will be written on the blackboard?

31. Prove that the number 3999991 is not prime.

32. (a) Find a seven-digit number with all its digits different, which is divisible by each of those digits.
    (b) Does there exist an eight-digit number with the same property?

33. We calculate the sum of the digits of the number 19^{100}. Then we find the sum of the digits of the result, et cetera, until we have a single digit. Which digit is this?

34. Prove that the remainder when any prime number is divided by 30 is either 1 or a prime number.

35. Does there exist a natural number such that the product of its digits equals 1980?

6 Digits

36. A natural number ends in 2. If we move this digit 2 to the beginning of the number, then the number will be doubled. Find the smallest number with this property.

37. Given a six-digit number $\overline{abcdef}$ such that $\overline{abc} - \overline{def}$ is divisible by 7, prove that the number itself is also divisible by 7.

38. Find the smallest natural number which is 4 times smaller than the number written with the same digits but in the reverse order.

39. A three-digit number is given whose first and last digits differ by at least 2. We find the difference between this number and the reverse number (the number written with the same digits but in the reverse order). Then we add the result to its reverse number. Prove that this sum is equal to 1089.

7 Which is greater?

40. Which number is greater: 2^{300} or 3^{200}?

41. Which number is greater: 31^{11} or 17^{14}?

42. Which number is greater: 50^{99} or $99!$?

43. Which number is greater: $888\ldots88 \times 333\ldots33$ or $444\ldots44 \times 666\ldots67$, where each of the numbers has 1989 digits?

44. Which type of six-digit numbers are their more of: those that can be represented as the product of two three-digit numbers, or those than cannot?

8 Word Problems

45. Two teams played each other in a decathlon. In each event, the winning team gets 4 points and the losing team gets 1 point, and both teams get 2 points in case of a draw. After all 10 events, the two teams have 46 points together. How many draws were there?[2]

46. If every boy in a class buys a muffin and every girl buys a sandwich, they will spend one cent less than if every boy buys a sandwich and every girl buys a muffin. We know that the number of boys in the class is greater than the number of girls. Find the difference.

47. 175 kiwis cost more than 126 passionfruit. Prove that you cannot buy three kiwis and one passionfruit for one dollar.

48. In a class, every boy is friends with exactly three girls, and every girl is friends with exactly two boys. It is known that there are only 19 desks (each holding at most two students), and 31 of the students in the class study French. How many students are there?

[2]From Mathematical Circles, the Russian Experience

49. Four friends bought a boat. The first friend paid half of the sum paid by the others; the second paid one third of the sum paid by the others; the third paid one quarter of what was paid by the others; and the fourth paid 130 dollars. What was the price of the boat, and how much did each of the friends pay?

50. The road connecting two mountain villages goes only uphill or downhill (never flat). A bus always travels 15 mph uphill and 30 mph downhill. Find the distance between the villages if it takes exactly 4 hours for the bus to complete a round trip.

9 Miscellaneous

51. Several identical paper triangles are given. The vertices of each one are marked with the numbers 1, 2, and 3. They are piled up to form a triangular prism. Is it possible that all the sums of the numbers along the edges of the prism are equal to 55?

52. Can one place 15 integers around a circle so that the sum of every 4 consecutive numbers is equal either to 1 or 3?

53. Find a thousand natural numbers such that their sum equals their product.

54. The numbers 2^{1989} and 5^{1989} are written one after another. How many digits in all are there?

55. A six-digit number is called *lucky* if the sum of the first three digits equals the sum of the last three. Prove that the number of "lucky" six-digit numbers equals the number of six-digit numbers with the sum of their digits equal to 27.