Euclid's Game, Part 2

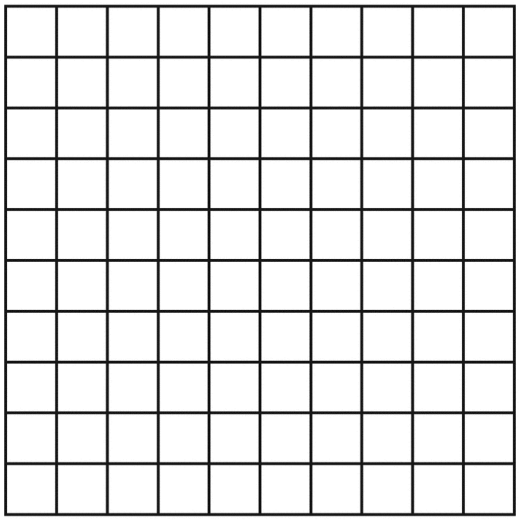
October 21, 2023

1 Euclid's Game - Avoid 0 Version

Last time, we played a version of Euclid's game where the goal was to reach the number 0. Today, we're going to play a version where the goal is to AVOID reaching the number 0. [1]

- Start with two numbers, e.g. 10 and 3.
- A move consists of taking away a positive multiple of the smaller number from larger number, so that the result is still positive.
- So from $(10, 3)$,
  - We can move to $(7, 3)$ by taking away 1×3 from 10.
  - Or we can move to $(4, 3)$ by taking away 2×3 from 10
  - Or we can move to $(1, 3)$ by taking away 3×3 from 10
- The player who cannot move without reaching 0 or a negative number loses. The last player to move wins.
- Here is a sample game: $(10, 3) \to (7, 3) \to (4, 3) \to (1, 3) \to (1, 1)$

1. For the game that starts with $(10, 3)$ and goes $(10, 3) \to (7, 3) \to (4, 3) \to (1, 3) \to (1, 1)$ and then the game is over because the next move would have to hit 0.
   (a) How many moves were there in this game?
   (b) Did the first player win or the second player win?
   (c) Could the other player have won, from the same starting position of $(10, 3)$ if they made better moves?
2. Which player would you want to be, for the starting position $(10, 3)$, if you can make any moves you like that follow the rules?
3. How could you play this game with a token on a grid like this one?



[1] This week's version of Euclid's game is thanks to Brittany Ohlinger.

2 Who Wins?

4. Try playing from the following starting positions. Does the first player or the second player have the advantage?

   (a) $(5, 1)$
   (b) $(5, 3)$
   (c) $(9, 2)$
   (d) $(4, 4)$

5. Play some more games with other starting positions.

3 Winning and losing positions

6. For which starting positions does Player 1 have a winning strategy – that is, Player 1 CAN win if they play the best moves, no matter what Player 2 does? We call these "winning positions".

7. For which starting positions will Player 1 definitely lose if Player 2 plays their best moves possible? We call these "losing positions".

8. Color winning positions green and losing positions red on the grid.

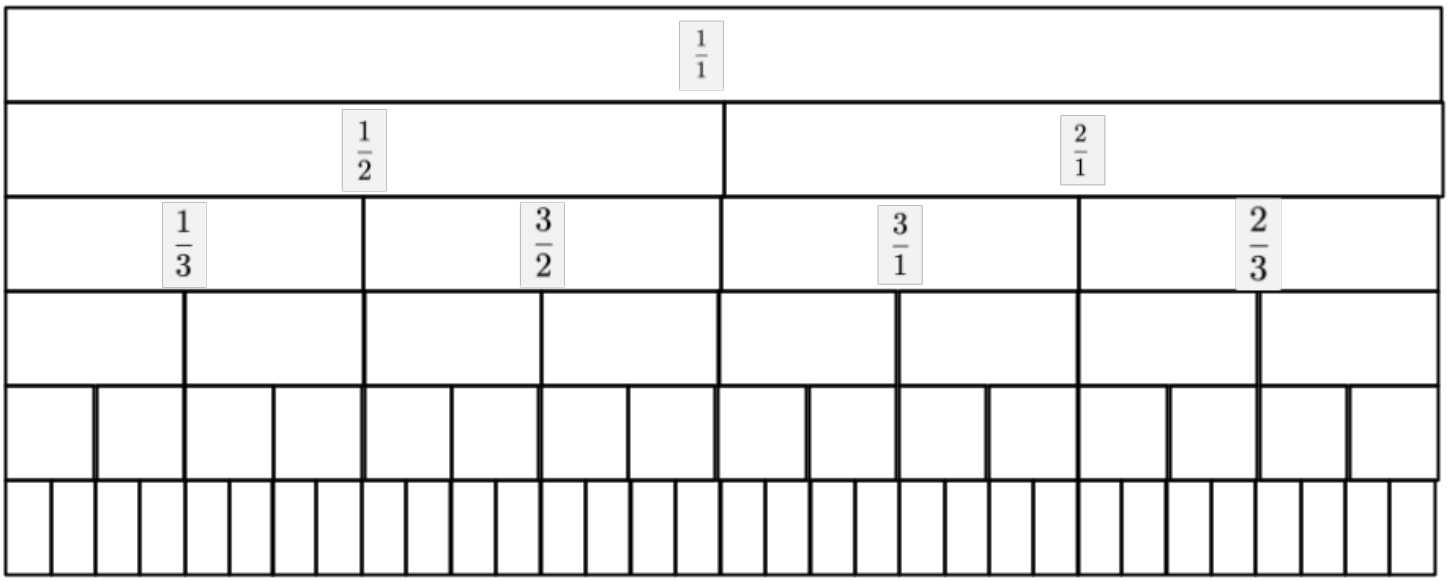
9. What patterns do you notice as far as which squares are green and which are red?

10. How can you use the chart to find the winning moves and win every game as long as you have a good starting position?

11. What are the slopes of the boundary lines?

4 Fraction Tree

Start with $\frac{1}{1}$ in the top box. Then build going down.



12. How is each fraction related to its "parent fraction" above it?
13. Fill in the rest of the fraction tree.

    We can represent the winning squares and losing squares for Euclid's game on the fraction tree. For example, position $(3, 2)$ is the fraction $\frac{3}{2}$ and position $(2, 3)$ is the fraction $\frac{2}{3}$.
14. Are there any positions for Euclid's game that don't appear on the fraction tree? Why?
15. Color the winning squares red and the losing squares green on the fraction tree.
16. What patterns do you notice? Can you explain why they happen?

5 Euclid's Game - Catch 1 Version

In this version of Euclid's Game, everything is the same except that a player wins by getting one of the numbers to be the number 1, instead of getting to the number 0 (the catch 0 version), or avoiding the number 0 (the avoid 0 version).

For example, starting at $(15, 4)$, a game might go like this:

$(15, 4) \to (7, 4) \to (3, 4) \to (3, 1)$ and then the game is over.

17. Play a few games for the Catch 1 version.

18. (a) Which starting positions are winning positions for the first player? That is, the first player can win if they make the best choices, no matter what the second player does? These are the "winning square".

    (b) Which starting positions are losing positions for the first player? That is, the first player will lose no matter what they do, if the second player makes the best choices possible. These are the "losing squares".

19. Color the winning squares green and the losing squares red.

20. Are there any squares that are neither winning squares nor losing squares? Which ones?

21. Why does the Catch 0 game always have a winner or loser, but the Catch 1 game does not?

22. Color the winning squares for the Catch 1 game on the fraction tree. Do you notice any patterns?

23. What other versions of Euclid's game could we play?

6 The Golden Ratio

The slope is the *golden ratio*, a famous number in math.

Here are a few of the many places that the golden ratio comes up:

Fibonacci sequence

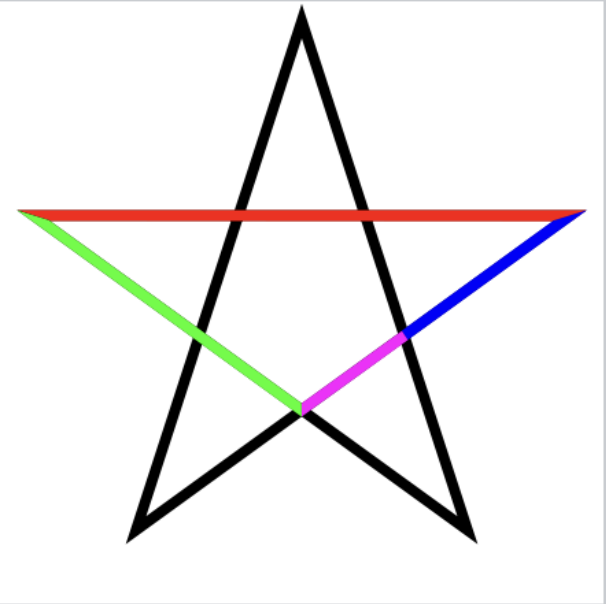
This list of numbers is called the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8 \cdots$$

To get the next number, add the two previous numbers together.

24. Write down the next 5 numbers in the sequence.
25. Use a calculator to divide each number in your sequence by the previous number and record the answer.
    (a) $\frac{1}{1} =$
    (b) $\frac{2}{1} =$
    (c) $\frac{3}{2} =$
    (d) $\frac{5}{3} =$
    (e) $\frac{8}{5} =$
    (f) keep going ....
26. What do you notice about these ratios? What number do they get close to?

Pentagram

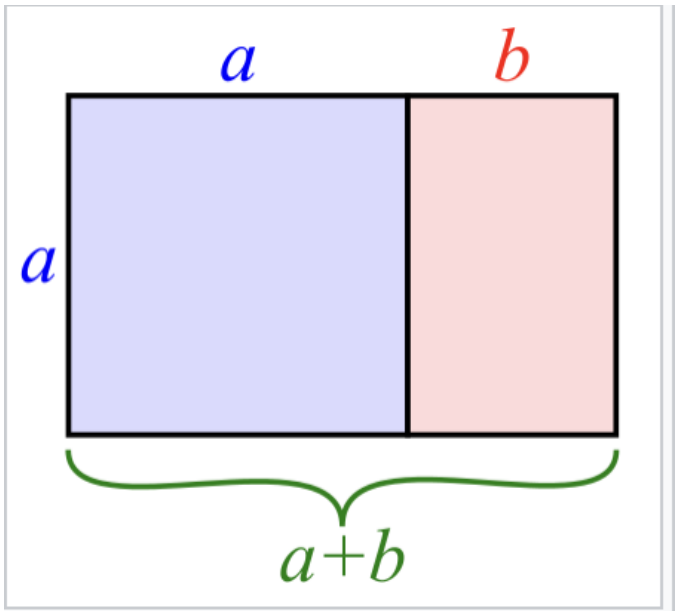


27. Measure the colored line segments carefully.
    (a) pink:
    (b) blue:
    (c) green:
    (d) red:

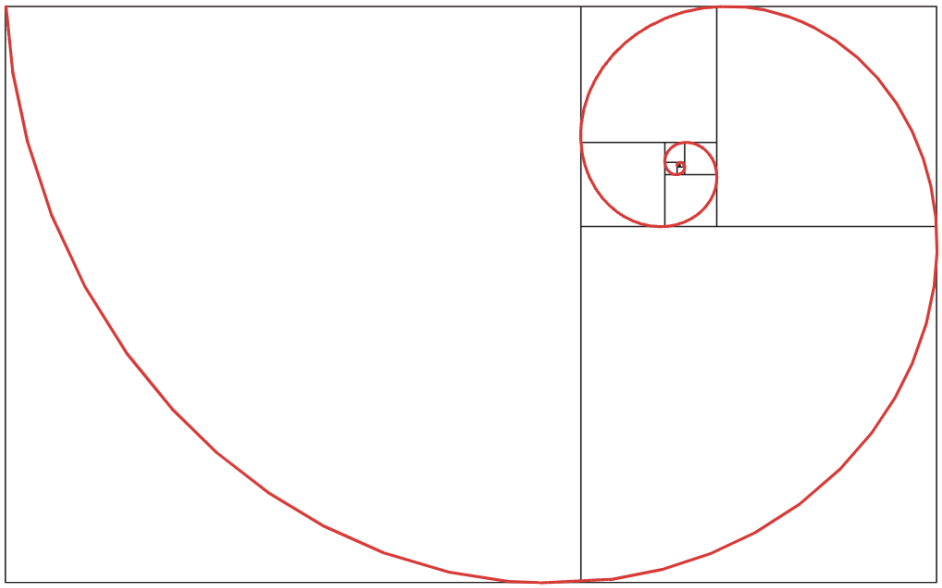
28. Divide the length of each colored line segment by the length of the previous one. What number are these ratios close to?

Golden Rectangle

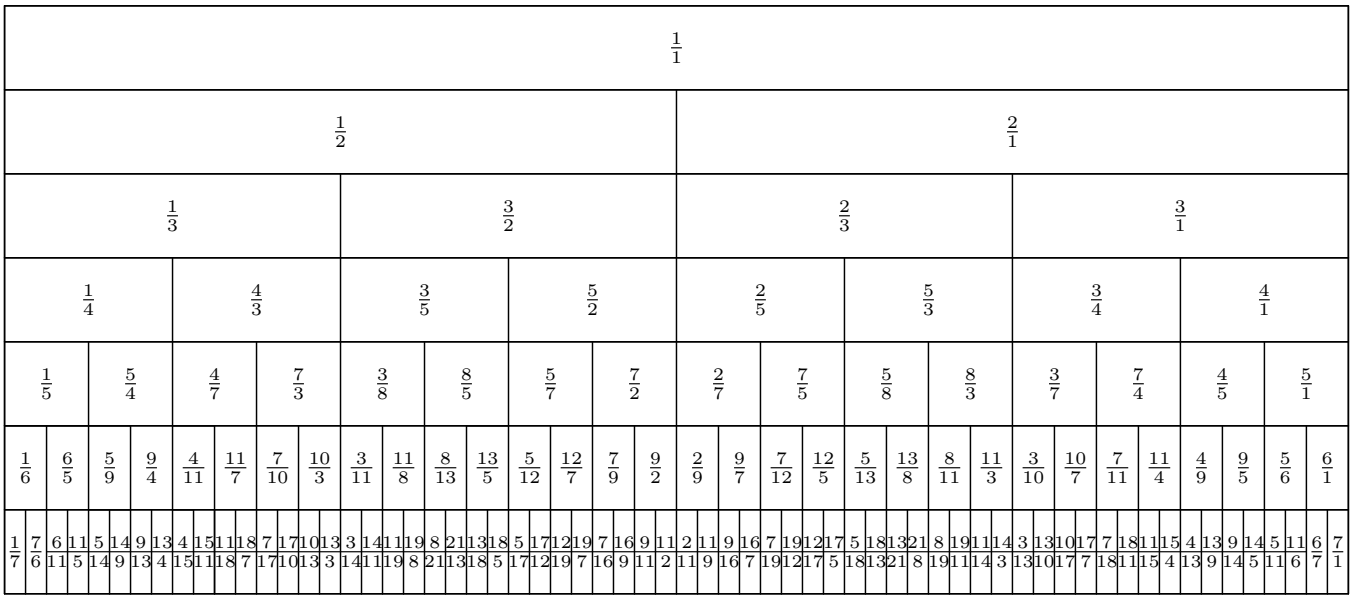
This large rectangle is called a Golden Rectangle. If we draw a square inside the rectangle, the leftover rectangle, turned sideways, has the same shape, that is, the same proportion between its length and width.



If we keep drawing squares, we get the Golden Spiral.



29. Measure the long side and the short side of the Golden Rectangle and divide the bigger side length by the smaller one. What number do you get?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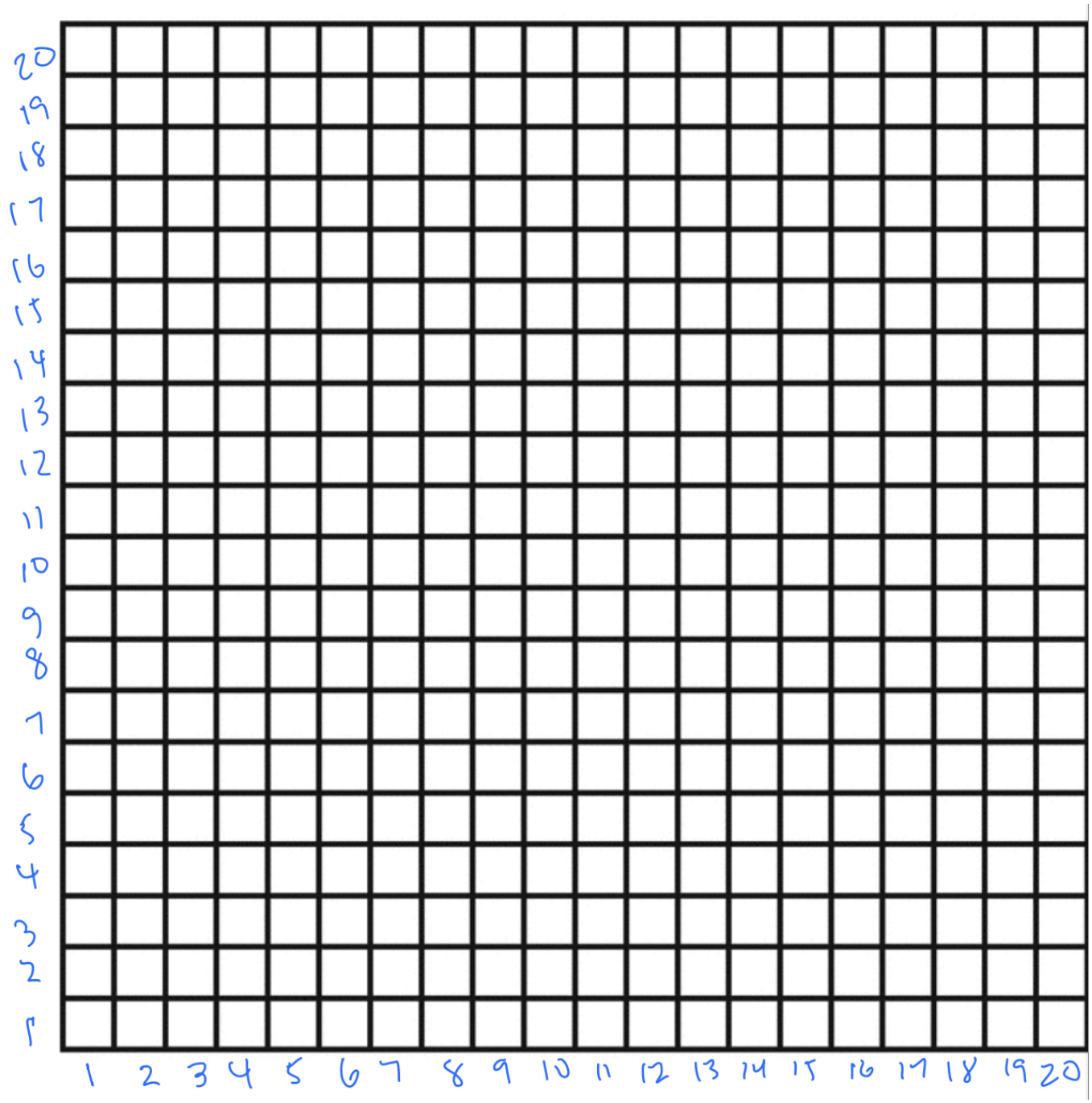
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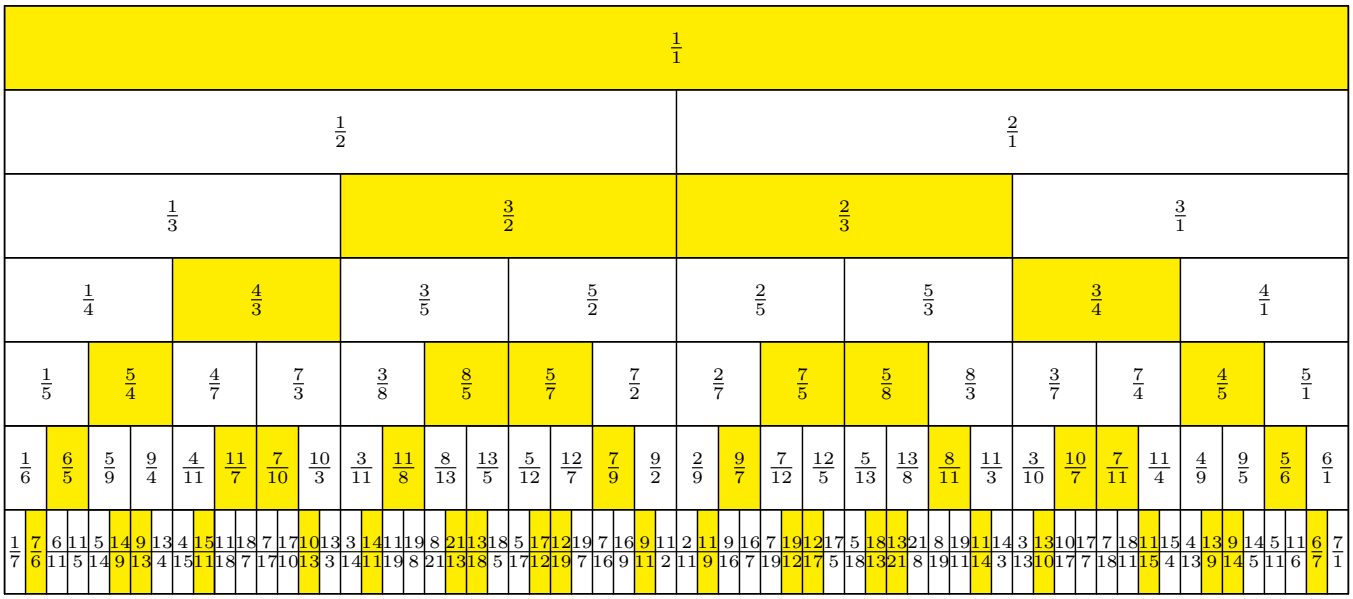
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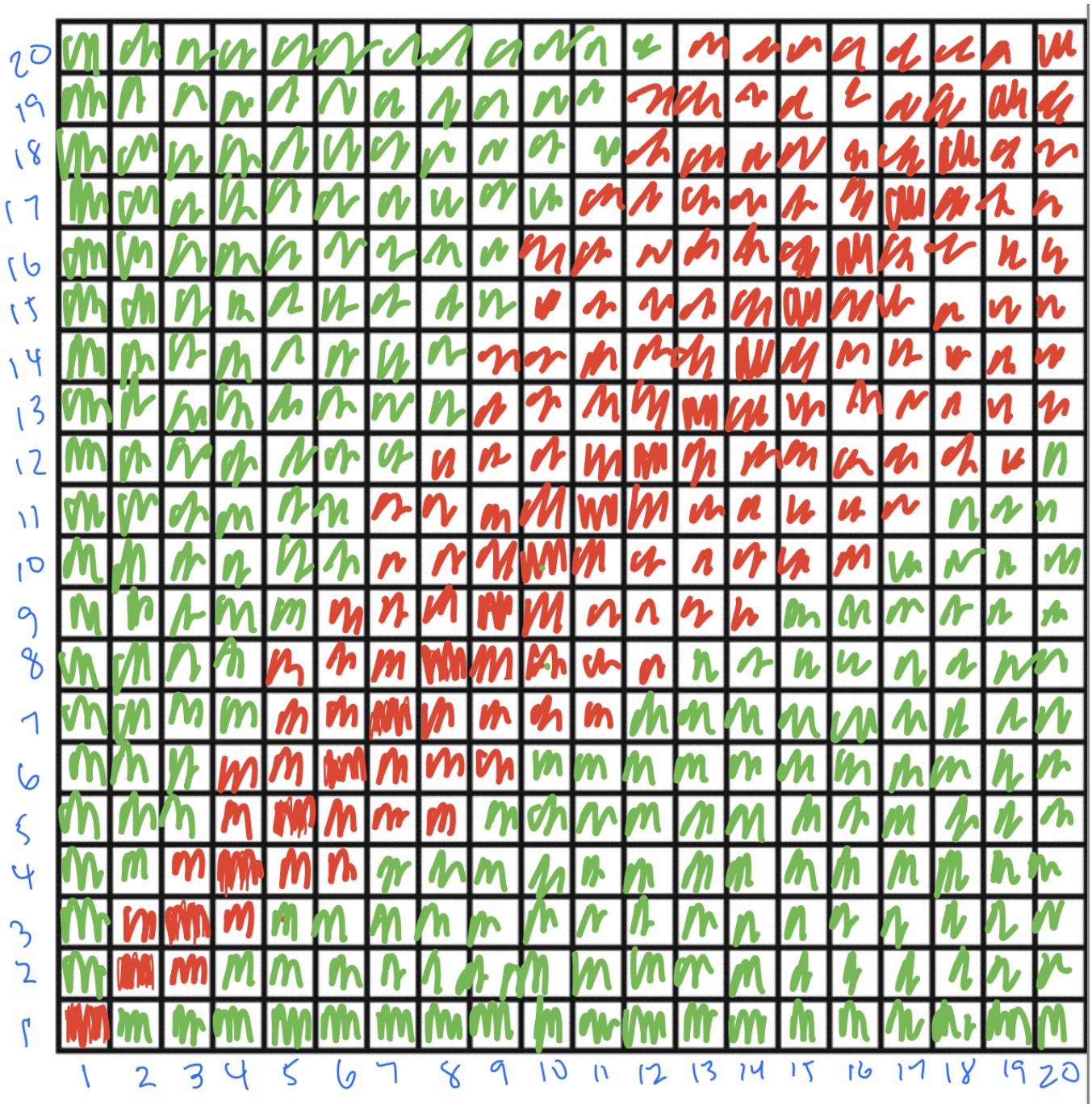
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