Chromatic Number *

Jan 28, 2023

1 Warm-up problems

1. Is it possible to paint every point of the plane using exactly two colors so that every line contains points of only one color?
2. Is it possible to paint every point of the plane using exactly two colors so that every line contains points of exactly two colors?
3. Is it possible to paint every point of the plane using exactly three colors so that every line contains points of exactly three colors?
4. Is it possible to paint every point of the plane using exactly three colors so that every line contains points of exactly two colors?
5. Is it possible to color each point on a circle either red or blue in such a way that no three points of the same color form an isosceles triangle? What if instead of just two colors you can use a thousand different colors?

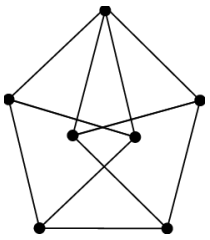
2 Chromatic Number

6. How many colors are needed to paint the real number line so that no two points a unit distance apart are painted the same color? (A "unit distance" means a distance of 1.)
7. How many colors are needed to paint the plane so that no two points a unit distance apart are painted the same color?

The minimal number of colors which are needed in order to paint all points of the Euclidean n-space (that is, space with n dimensions) in such a way that no two points a unit distance apart are of the same color is called the chromatic number of the space; it is usually denoted by $\chi(n)$.

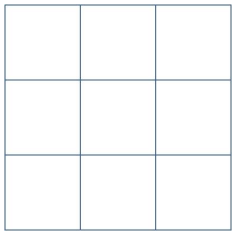
*Problems from Tatiana Shubin

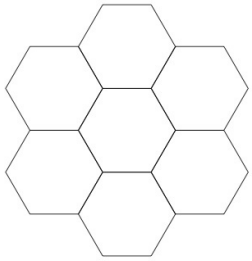
8. In the graph below (called the Moser Spindle), the length of each edge is 1. Is it possible to paint the vertices of the graph using three different colors so that no two adjacent vertices are painted the same color? (Note: vertices are called adjacent if they are connected by an edge.)

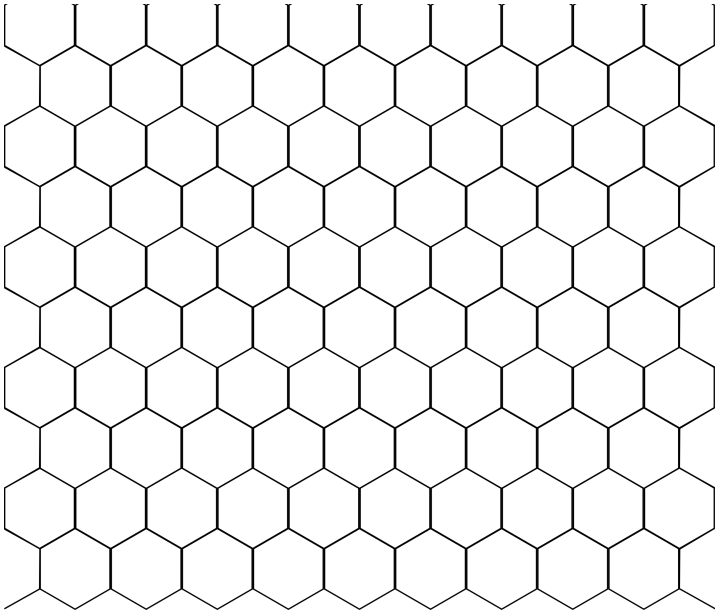


How does painting Moser Spindle help us to decide what are possible values for $\chi(2)$?

9. Now let's look at the following diagrams and see how they, too, help us narrow in on $\chi(2)$:







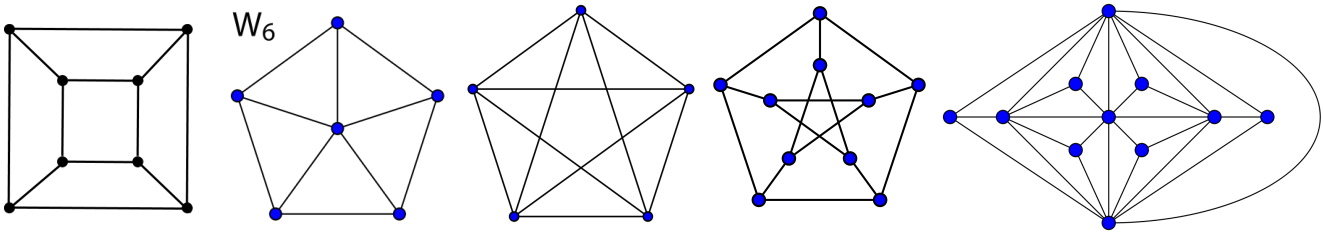
10. What is the chromatic number of 3-space? What lower bounds and upper bounds can you find?

3 Graph Coloring

Note to self: we didn't get to this part.

A *graph* is a collection of vertices (points) and edges (line segments, which can be curved). The edges connect vertices.

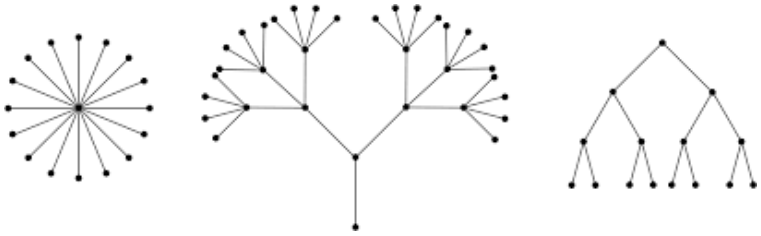
Here are some examples of graphs.



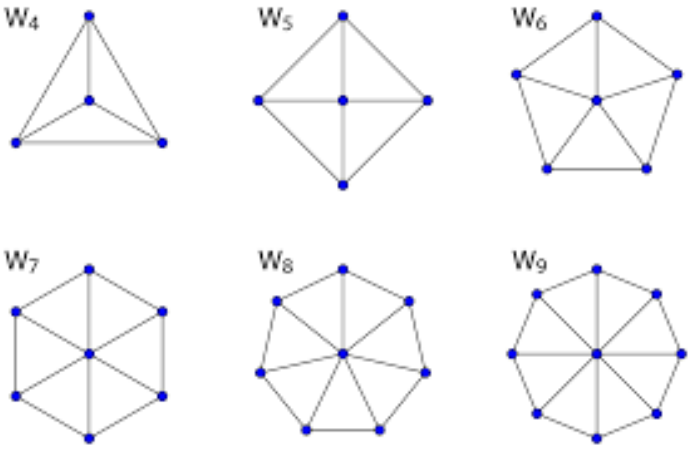
Notice that in the third and fourth graphs, the edges cross, but don't have a vertex there. That is allowed.

The *chromatic number* of a graph is the smallest number of colors you need, in order to color all the vertices such that no two adjacent vertices have the same color. Vertices are called *adjacent* if they have an edge between them.

11. Find the chromatic number of the graphs above.
12. Find the chromatic number of these tree graphs.



13. What are the chromatic numbers of these wheel graphs?



14. A graph is called *planar* if it can be drawn so that no edges cross. Can you find a planar graph with chromatic number 2? 3? 4? 5?