A few of these problems come from *The USSR Olympiad Problem Book*, by D.O. Shklarsky et al. There are some really good problems in this book!

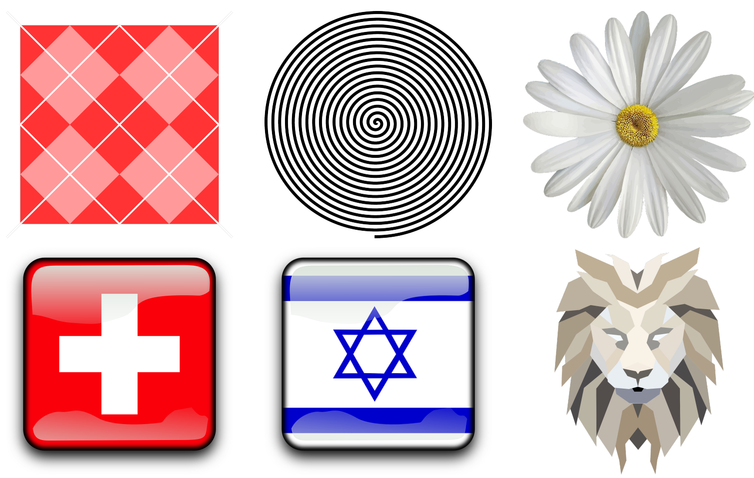
Symmetry: Part I

For the next few weeks we are going to study symmetry in mathematics. Symmetry is one of the most fundamental properties of objects to study in math. By studying symmetry on many different levels, we will be able to understand and solve problems in elegant and insightful ways.

The first thing we need to do is develop a good sense of what symmetry actually is. You may already have a general idea of when things are symmetric, but in math we often want to write down exactly what we mean by a word.

1 Warm Up

1. What symmetries do the following objects have?



2. We might call the symmetry above *geometry symmetry*. In what ways are the following expressions symmetric?

$$\dots ABBAABBA \dots$$

$$f(x, y, z) = xyz$$

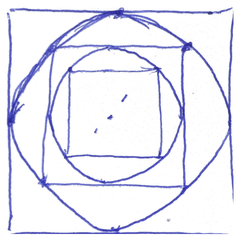
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 0 \end{bmatrix}$$

3. Write down, in your own words, what it means for an object to be symmetric:

2 Problems

Some of the following problems have long brute-force solutions. Try to use symmetry to your advantage instead.

1. A circle is inscribed inside a square of length 1. A square is then inscribed in this circle, and then a circle inside this square. This pattern continues forever. What is the area of all the squares combined?



2. Find all the solutions to the following system of equations:

$$x + 2y + 3z = 30$$
$$2x + 3y + z = 30$$
$$3x + y + 2z = 30.$$

3. Which of the expressions,

$$(1 + x^2 - x^3)^{1000} \text{ or } (1 - x^2 + x^3)^{1000},$$

will have the larger coefficient for x^{20} after expansion and collecting terms.

(Hint: Try smaller powers).

4. (a) Prove that in the product

$$(1 - x)(1 + x),$$

after multiplying and collecting terms, there does not appear a term in x of odd degree. What is this multiplication called?

(b) Do the same for

$$(1 - x + x^2)(1 + x + x^2)$$

and

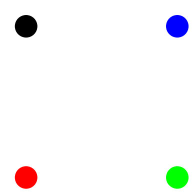
$$(1 - x + x^2 - x^3)(1 + x + x^2 + x^3).$$

(c) Prove the same thing for

$$(1 - x + x^2 - x^3 + \ldots - x^{99} + x^{100})(1 + x + x^2 + \ldots + x^{99} + x^{100}).$$

(Hint: Multiplying this out by hand will not be fun. What kind of symmetry can you apply to an expression like this?).

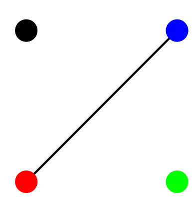
5. Suppose that we have 4 dots below:



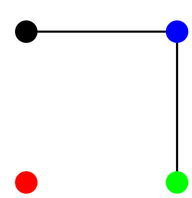
(a) List some symmetries of this object. Describe them in detail, including the operation and the fixed property.
(b) Suppose that we are allowed to swap the dots around in any way we want. Describe the symmetry here.
(c) How many different ways can we swap the dots?

6. Do parts (b) and (c) from part 5, but instead with these objects where when you switch the dots, the lines must stay intact. Do you notice any patterns, relationships with these symmetries?

(a)



(b)



(c)