CHMC Advanced: Non-transitive dice

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1 Introduction

In this worksheet, we'll explore the fascinating world of non-transitive dice. These dice are constructed by taking a cube and labelling each face with a non-negative integer, though the dice isn't required to have every integer $1, 2, 3, \ldots, 6$ present. By labelling the dice in the right ways, you get a situation where you can play "rock-paper-scissors" with dice (in an average sense).

2 3 dice

- Look at the red, blue, and olive dice in the handout (the 3-dice set). What are the values on each of the dice?
- Which dice, on average, beat which dice? How do you know? (What does it mean to "beat on average"?)
- Suppose that, instead of playing with one dice of each colour, you play with two dice of the same colour. Which pairs beat which (on average)?
- How do the names of the dice relate to which beat which?

3 5 dice

- Look at the red, blue, olive, yellow, and magenta dice on the handout (the 5-dice set). What are the values on each of the dice?
- Which dice, on average, beat which dice? How do you know?
- Suppose instead of playing with one dice of each colour, you play with two dice of the same colour. Which pairs beat which (on average)? Are the changes here the same as the changes you noticed for the "3 dice"?
- How do the names of the dice relate to which beat which?

4 Some related problems

- Here's a set of four dice:
  - Blue: $3, 3, 3, 3, 3, 3$;
  - Magenta: $2, 2, 2, 2, 6, 6$;
  - Olive: $1, 1, 1, 5, 5, 5$;
  - Red: $0, 0, 4, 4, 4, 4$.

Are these dice non-transitive? Do they have the same flipping property that the "3 dice" and "5 dice" had?

- Here's a set of three dice that utilize famous mathematical constants:
  - A: $1, 1, 1, 1, 1, \pi$;
  - B: $\Phi, \Phi, \Phi, e, e, e$;
  - C: $0, \phi, \phi, \phi, \phi, \phi$.

  Here, $e \approx 2.7182$ is Euler's constant, $\pi \approx 3.1415$ is π, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.6180$ is the golden ratio, and $\Phi = \frac{-1+\sqrt{5}}{2} = \frac{1}{\phi} \approx 0.6180$ is the golden ratio conjugate. Do these three dice form a non-transitive set?

- Consider the following 7 dice:
  - Dice 1: 10, 7, 16, 7, 10, 16;
  - Dice 2: 13, 5, 15, 5, 13, 15;
  - Dice 3: 9, 3, 21, 3, 9, 21;
  - Dice 4: 12, 1, 20, 1, 12, 20;
  - Dice 5: 8, 6, 19, 6, 8, 19;
  - Dice 6: 11, 4, 18, 4, 11, 18;
  - Dice 7: 14, 2, 17, 2, 14, 17.

  Are these dice non-transitive? They have another special property when it comes to three player games: suppose your two opponents pick their dice first; can you pick a dice that will beat both opponents simultaneously (on average)?

- Can you construct a non-transitive set of dice using negative numbers on some of the dice faces?

5 References

- https://singingbanana.com/dice/article.htmFN14