CHMC Advanced Section: Partitions

February 22, 2020

1 Introduction

By a partition, we mean a way to decompose/tile/tessellate space based off of some kinds of rules. The rules we'll focus on are based on distances to a point, and areas of each region of the partition.

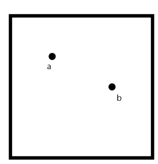
These kinds of partitions show up in many areas of both pure and applied math. Moreover, the efficient and fast computation of some of these partitions by computers are active areas of research today.

2 Nearest neighbor partitions

The first kind of partition we'll explore is based off of the nearest neighbors to points in a space.

2.1 Partitions on a square

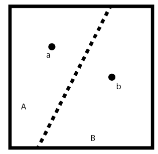
Suppose we're given a square, and two points in the square:



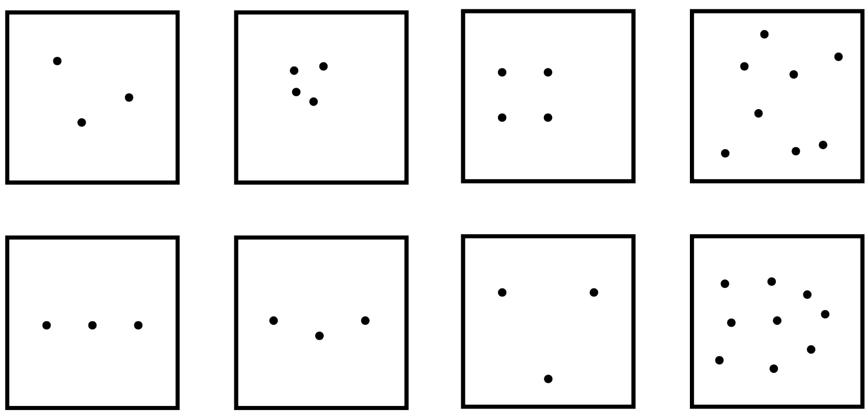
Suppose we want to split the square into two regions A and B such that points in A are closer to a, and points in B are closer to b. Here's one way to do this:

We'll call such a decomposition a **nearest neighbor partition**.

Exercise 2.1 Is this the only partition that works for these two points? Why or why not?



Exercise 2.2 What are the partitions for the following point configurations? Sketch them.



Exercise 2.3 One way to construct a nearest neighbor partition is by looking at growing disks centered at each point. If you place a disk at each point, and let the radii of all of the disks grow at the same rate, then eventually each disk will intersect some other disk; how do these intersections relate to the nearest neighbor partitions you sketched above?

Exercise 2.4 In general, is the nearest neighbor partition for a given set of points uniquely determined?

Nearest neighbor partitions are also known as **Voronoi decompositions**, and appear many places in nature and the other sciences. For example, the source of the 1854 cholera outbreak in London was detected using a sort of nearest neighbor partition, where the points themselves were water pumps.

One important note: the regions of a nearest neighbor partition won't, in general, have the same area as any of the other cells. We'll look at equal area partitions later.

Exercise 2.5 The cells in a nearest neighbor partition are always **convex**, meaning that if you take two points in the same cell and draw a line between them, the line will never leave the cell. Why is this true?

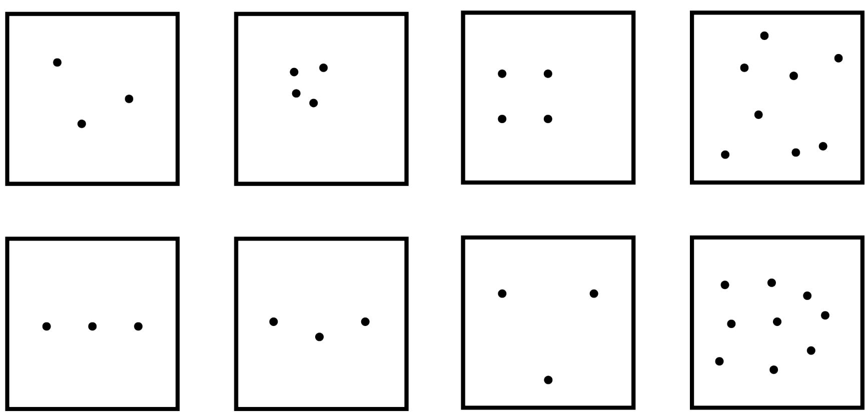
The distance we've been using is the Euclidean distance, where the distance between two points a and b is computed by taking the length of the straight line segment connecting a to b. We can certainly consider partitions with respect to different distances, which the next few exercises explore.

Exercise 2.6 Suppose we measure the distance between two points $a = (a_1, a_2)$ and $b = (b_1, b_2)$ by the formula

$$d(a, b) = \max\{|a_1 - b_1|, |a_2 - b_2|\}.$$

Sketch the disk of radius 1 centered at $(0, 0)$. Does this "disk" look like a disk?

Exercise 2.7 For the following point configurations, sketch the nearest neighbor decomposition using the distance from the last exercise. How do your partitions change?



Exercise 2.8 We've been looking at points in squares, and building the nearest neighbor partitions with respect to the square. What if the points were in a plane? How would your partitions change (for whichever distance you want to use)?

Exercise 2.9 What are the areas for each of the cells in a nearest neighbor partition? What needs to happen for a cell to have finite area? What is the minimum number of points needed to ensure you have a finite area cell?

The nearest neighbor partitions also have a "dual" structure, which the next exercises illustrate.

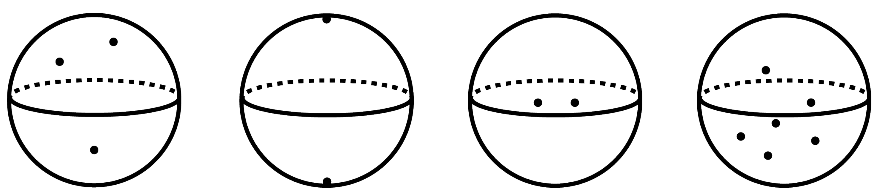
Exercise 2.10 For each of the partitions you've constructed above, draw edges between two points if their cells intersect. We'll call these edges **nearest neighbor edges**.

Exercise 2.11 Is it possible for a nearest neighbor edge to cross through a cell besides the two cells of the points the edge connects? When does this happen?

2.2 Partitions on a sphere

We can play the same game from above, now with spheres! Here, the distance used is computed by drawing a great arc between two points, and then taking the arc length of that circle. For example, on a sphere of radius 1, the distance between the north and south pole is π.

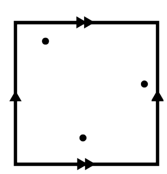
Exercise 2.12 For the following point configurations, sketch what the nearest neighbor partitions must be. Feel free to use pins and foam balls to help visualize your sketches, but don't draw on the foam balls.



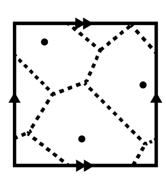
2.3 Partitions on a torus

The last kind of space we'll look at is the torus, which can be constructed by taking a square and adjoining opposite sides in such a way so that: 1) you get a cylinder, and then 2) you get a donut.

Consider these three points on a torus:

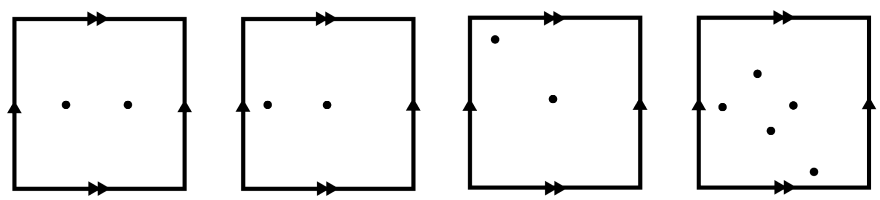


Here's an example of a nearest neighbor partition for the same 3 points:



One "quick" way to get this partition is to make 9 copies of your torus, and then construct the partition for the nine squares counted as one; in practice, you just need to draw enough partitions so that every point in the central cell is in some nearest neighbor cell. Then, project all of the partition lines onto a single region for the torus, and that's your partition! Figure 1 at the end of this worksheet illustrates this "quick" way for the 3 points above.

Exercise 2.13 What are the partitions for the following point configurations? Sketch them. Remember that we're working on a torus, i.e. the sides are identified.



3 Equal area partitions

The second kind of partition we'll look at today is something called an **equal area partition**. As the name suggests, we're interested in partitioning a space into cells that each have equal area. We're also going to assume that our cells are all convex.

3.1 Partitions on a square

Exercise 3.1 How many ways are there to partition a square into 2 cells with equal area? 3 cells? n cells? Sketch possible partitions for 3 and 4 cells.

The **diameter** of a cell in a partition is the maximum distance between any two points in a partition. Since in general there are many ways to partition a square into cells of equal area, we're going to impose some constraints on the diameter of the cells that make up a partition. Assume that the vertices of the square are the points $(0,0), (1,0), (0,1)$, and $(1,1)$.

Exercise 3.2 What is the maximum diameter of any cell for an equal area partition having 2 cells?

Exercise 3.3 Is it possible to have an equal area partition of 4 cells, with diameter no less than $\frac{\sqrt{2}}{4}$? What is the minimum diameter that lets you have an equal area partition of 4 cells?

Exercise 3.4 In general, for an equal area partition with n cells, what is the minimum diameter you'll need for your cells?

3.2 Volume partitions

Now, suppose the square is a sandpit. If the sand was equally spread all around the square, then a partition of the sandpit into regions with equal volume would correspond to an equal area partition. Sandpits don't usually have this property, however, and so an interesting question is then: how should we partition the sandpit so that each cell has an equal volume of sand?

Exercise 3.5 Suppose the square has a lot of sand in the center, and little sand towards the boundary (like a pyramid or a hill). Sketch what an equal volume partition with 5 cells might look like. Do the cells have the same sizes?

Exercise 3.6 Does your partition from the last exercise correspond to a nearest neighbor partition for some collection of points? Why or why not?

Exercise 3.7 Try the same exercise, now with two bumps/hills/pyramids in the square, again with 5 cells; try it again with 8 cells. Which partition is easier to sketch?

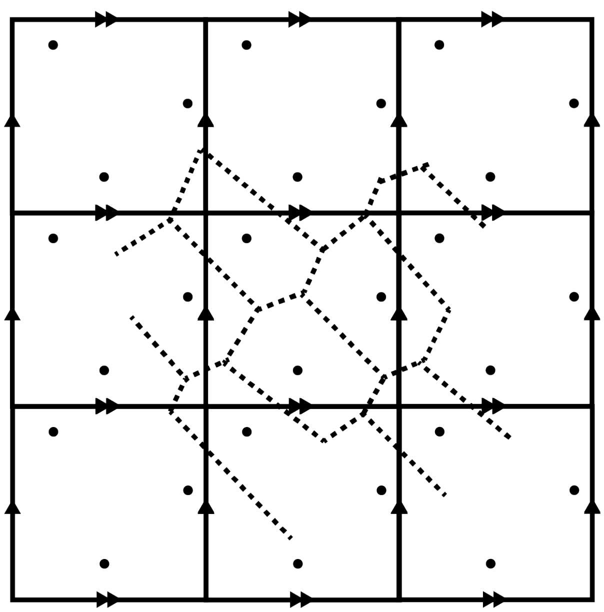


Figure 1: Torus partition example.