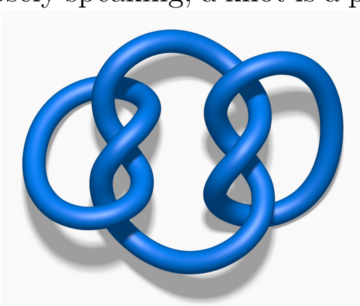
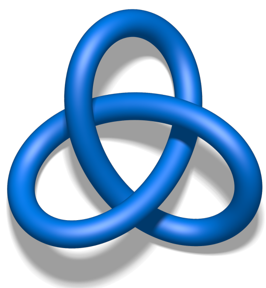
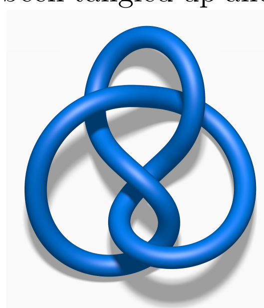
Knots

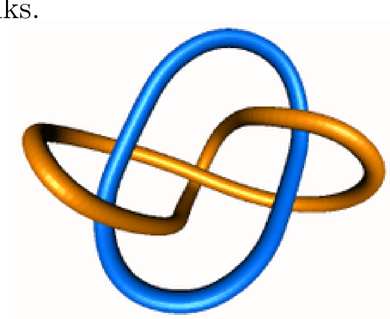
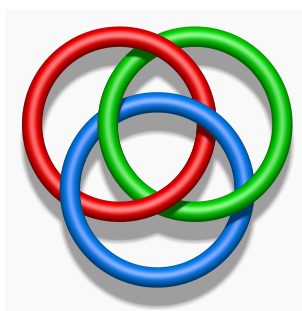
From Maia Averett and *A Decade of the Berkeley Math Circle, Volume 2*

1 Warm-Up

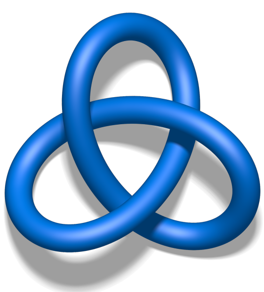
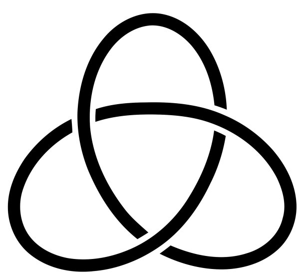
Loosely speaking, a knot is a piece of string that has been tangled up and then had the ends fused together.

When the knot is made from two or more loops of string, it is called a link. Here are some examples of links.

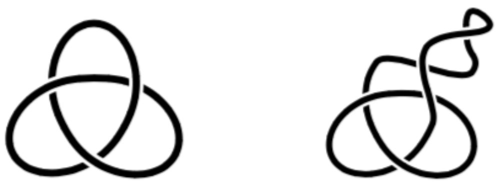
Knots can be represented by ”knot diagrams”. For example, the knot below at left can be represented by this knot diagram at right, where the gaps represent undercrossings.

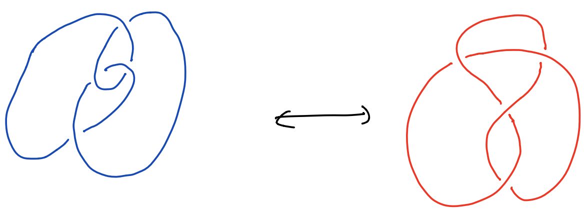
Here are some general questions we will explore today:

- When should we consider two knots to be the same?
- How can we tell different knots apart?
- How can we tell if a knot is knotted or not?

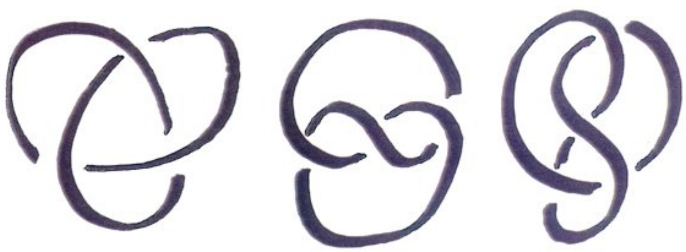
1. In what sense are the following knots equivalent?



2. Are these knots equivalent?

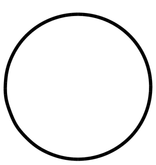


3. Which of these knots, if any, are equivalent?

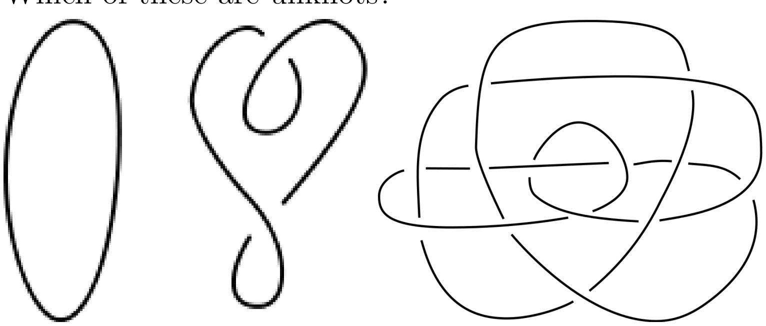


(a) (b) (c)

Definition: An *unknot* is a knot equivalent to this one:

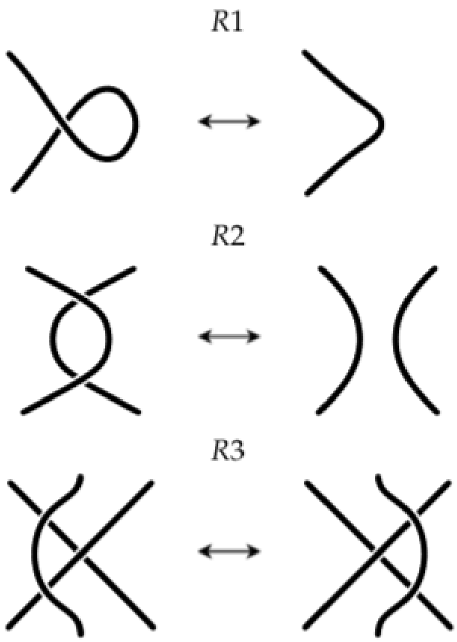


4. Which of these are unknots?



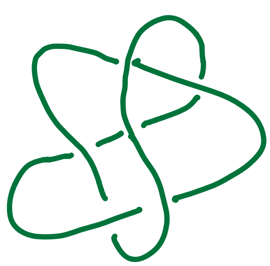
2 Reidemeister Moves

Definition: A Reidmeister move is one of the following three moves:



Each move has several variations. For example, in a type R3 move, the strand might be entirely under instead of over the crossing.

5. Use a sequence of Reidmeister moves to get from this knot diagram to the standard diagram for the unknot.



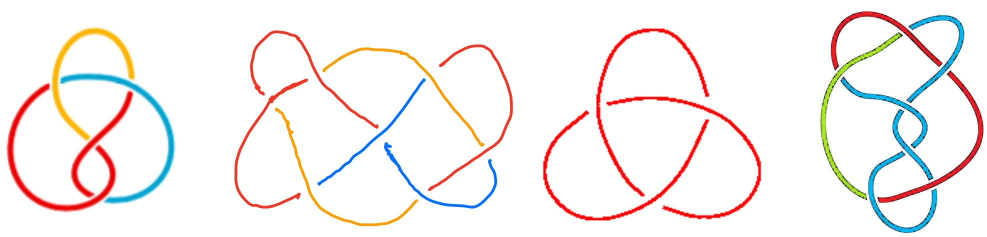
6. Suppose we can get from one knot diagram to another by a sequence of Reidmeister moves. Are the two knots represented by the two knot diagrams necessarily equivalent?

7. Suppose we have two knots that are equivalent, but are represented by two different knot diagrams. Can we get from one knot diagram to the other by a sequence of Reidmeister moves?

3 Knot Coloring

8. The following knot colorings are legit ”tricolorings”.

The following knot colorings are not legit tricolorings.

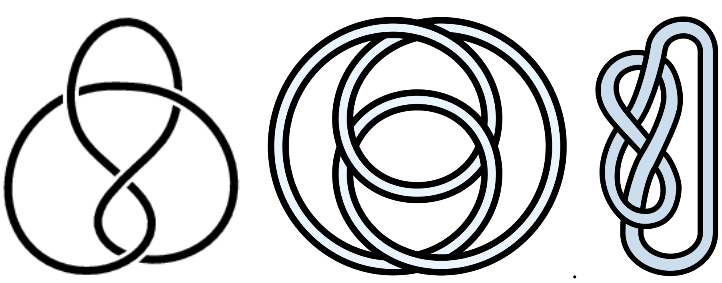


What are the rules are for a knot coloring to be a legit tricoloring?

9. Suppose you and your friend have two different knot diagrams for the same knot. If your friend can tricolor hers, does that necessarily mean that you can tricolor yours?

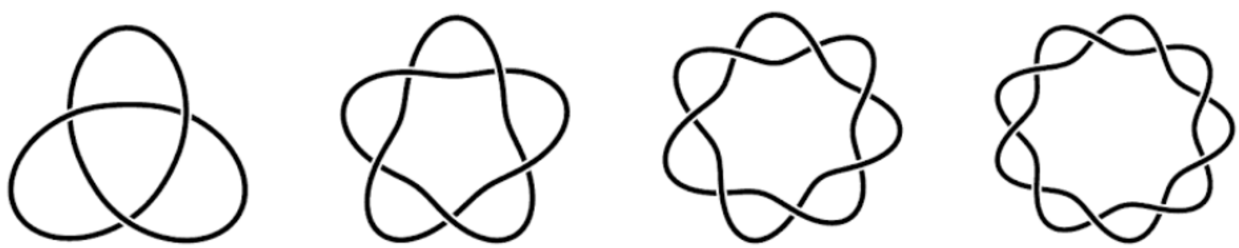
In case you want to experiment, here are some different knot diagrams for the trefoil knot.

And here are some different knot diagrams for the figure eight knot.



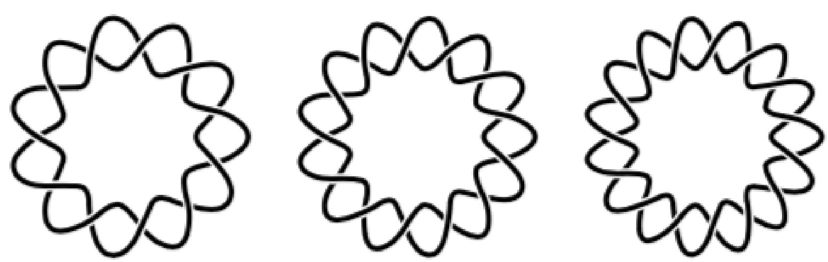
10. Prove that tricolorability is a "knot invariant" – that is, it remains unchanged under Reidemeister moves: if one knot diagram can be tricolored, then another knot diagram of the same knot that differs by a Reidemeister move can also be tricolored.

11. Which of these knots are tricolorable? They are the trefoil, the 5_1 knot, the 7_1 knot, and the 9_1 knot (why do you think they have those names?). What do you notice about your answers? Make a conjecture and explain your reasoning.

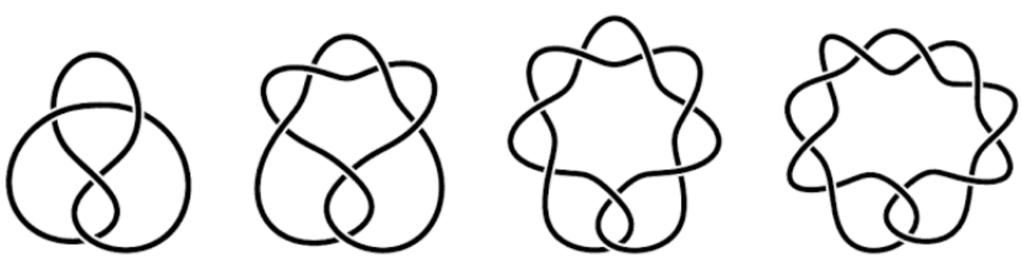


Note that to prove that a knot is tricolorable, you just need to demonstrate a tricoloring. To prove that a knot is NOT tricolorable, you have to come up with a logical argument to show that no tricoloring at all could be possible.

12. Here are the 11_1, 13_1, and 15_1 knots.



13. Here are the 4_1, 6_1, 8_1, and 10_1. Which are tricolorable? Make a conjecture and explain your reasoning.



Draw the 12_1 knot and check your conjecture.

14. What does tricolorability tell you about the figure 8 knot, the trefoil, and the unknot?

4 Counting Tricolorings

15. We can also think about not only whether or not a knot is tricolorable, but how many possible tricolorings there are. Let's write $\tau(K)$ for the number of tricolorings of a knot K.

16. Prove that $\tau(K)$ is a knot invariant (i.e. it does not change when you perform Reidemester moves).

17. Compute $\tau(K)$ for the trefoil, the figure 8 knot, and the square knot (shown below as the connect sum of the trefoil and its mirror image). Conclude that these are all different knots! Do you notice a pattern? Can you explain why you see that pattern?

18. The previous problem shows that τ is a more refined invariant than the simple yes-no of tricolorability: it can distinguish between the trefoil and the square knot even though both are tricolorable.

5 Colorings and Connected Sums

The connected sum $K_1\#K_2$ of two knots K_1 and K_2 is formed by erasing a little piece of each knot and then connecting the loose strands together, as shown in an example below. This example takes a copy of the right-handed trefoil and the left-handed trefoil and forms their connected sum, which goes by the name of the square knot.

K_1 K_2 $K_1\#K_2$

If K_1 and K_2 are tricolorable, is $K_1\#K_2$ tricolorable?

19. Find a formula that relates $\tau(K_1$, $\tau(K_2)$, and $\tau(K_1\#K_2)$.