CHMC Advanced Group: Derangements

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1 Introduction

In this worksheet, we will investigate a combinatorial object called a derangement. These are interesting objects in their own right, however the tools we can use to study them are also worthwhile concepts for exploration. We will introduce the concept of permutations in the opening. Along the way, the Principle of Inclusion-Exclusion and recursive sequences will prove to be profitable tools for investigating derangements.

2 Derangements

In this section, we introduce the notion of a derangement and work to understand some interesting facts about them.

Suppose there are $n > 1$ distinct people, $\{p_1, p_2, \ldots, p_n\}$ and each person is assigned a person in this list a letter. For example, let $f : \{p_1, p_2, \ldots, p_n\} \to \{p_1, p_2, \ldots, p_n\}$ give the assignment, so if $f(p_1) = p_3$ this means that person 1 sends a letter to person 3. We require that every person sends a letter and every person receives a letter; this means there are not distinct indices $1 \leq i, j, k \leq n$ for which $f(p_i) = p_k = f(p_j)$. Such a function is called a *permutation* on n objects.

Exercise 2.1 How many permutations are there for $n = 3$? How about $n = 4$? And $n = 5$? What about general n?

A *derangement* is a permutation for which $f(p_i) \neq p_i$ for each $i = 1, \ldots, n$; that is, everyone sends a letter to someone other than themselves.

A useful way to represent a permutation is in the following manner: Start with an open left parenthesis (. Next, write p_1. Following p_1, write $f(p_1) = p_k$. Then write $f(p_k) = p_j$. Continue doing this until you loop back to p_1, closing with a right parenthesis). Once this happens, start with a left parenthesis (and the next p_i not already recorded. Continue this process until all indices are used. This is called *cycle notation*.

For a worked example, let $n = 5$ and define f by $f(1) = 3,\ f(2) = 5,\ f(3) = 4,\ f(4) = 1,\ f(5) = 2$. This we write as (1 3 4)(2 5).

Exercise 2.2 Write the following permutations in cycle notation:
1. $n = 5,\ f(1) = 3,\ f(2) = 4,\ f(3) = 5,\ f(4) = 1,\ f(5) = 2$.
2. $n = 5,\ f(1) = 1,\ f(2) = 4,\ f(3) = 5,\ f(4) = 2,\ f(5) = 3$.

In practice, if the cycle notation of a permutation includes a single number in parentheses, this can be omitted; it is understood in this case the the permutation maps that element to itself. For example, for $n = 4$, the permutation (1 3)(2)(4) can be rewritten as (1 3). The empty cycle, denoted () represents the permutation that fixes every object.

Exercise 2.3 Let σ be a permutation on n objects. If σ is a derangement, how many numbers appear in its cycle notation representation?

Exercise 2.4 For $n = 6$, which of the following are derangements?
1. (1 2 4)(3 5).
2. (1 5 2 6 3 4).
3. (1 2)(3 4)(5 6).
4. (1 5).

Now, we list out the possible permutations for a fixed integer n and determine which are derangements.

Exercise 2.5 List out all of the permutations on $n = 1$ object. In this list, is there a derangement?

Exercise 2.6 List out all of the permutations on $n = 2$ objects. In this list, which of the permutations are derangements?

Repeat the process above for $n = 3, 4$.

3 Principle of Inclusion-Exclusion

We now consider a particular useful way of counting. To begin, define a *set* to be an unordered list of objects. Any duplicates in the list are removed so there are no repeats. The objects in the set are called *elements* of the set. Examples of sets are:

Days of the week = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}, Primary Colors = {Red, Yellow, Blue}, Even integers = $\{\ldots, -4, -2, 0, 2, 4, \ldots\}$.

If A and B are given sets, the *union* $A \cup B$ of A and B is the set consisting of objects, each of which is an element of A or B (or both). The *intersection* $A \cap B$ of A and B is the set consisting of objects, each of which is an element of both A and B. We can extend the notion of union to more than two sets by setting the union of sets $A_1, A_2, \ldots, A_k$ to be the set of objects, each of which is an element of one of the A_i. Similarly, the intersection of sets $A_1, A_2, \ldots, A_k$ is the set consisting of objects each of which are in every A_i.

If A is a set, it is called *finite* if the number of objects in A is finite; it is infinite otherwise. We denote by $|A|$ the *cardinality* of A, that is, the total number of objects. If it is a finite number, then A is a finite set; if A is not finite, then we say A has infinite cardinality.

One question to ask is given two finite sets A and B, what is the value of $|A \cup B|$?

Exercise 3.1 Let A be the set consisting of positive even integers less than 10 and B the set of positive integers less than 6. What is $A \cup B$? How about $A \cap B$? Can you relate the values of $|A|$, $|B|$, $|A \cup B|$, and $|A \cap B|$?

Exercise 3.2 Repeat the previous exercise with A consisting of even positive integers less than 10, B consisting of odd positive integers less than 10.

Exercise 3.3 Conjecture a formula that expresses $|A \cup B|$ in terms of $|A|$, $|B|$, and $|A \cap B|$ for finite sets A and B.

Now, suppose instead of two sets we consider a third set C.

Exercise 3.4 Keep the sets A and B the same as in exercise 3.1. Let C be the set consisting of positive even integers greater than 4 and less than 12. Determine the value of $|A \cap B|$, $|A \cap C|$, $|B \cap C|$, and $|A \cap B \cap C|$. Can you relate these to $|A \cup B \cup C|$?

Exercise 3.5 Do the same as above with exercise 3.2, with $C = \{1, 2, 3, 4\}$.

Exercise 3.6 Conjecture a formula that expresses $|A \cup B \cup C|$ in terms of $|A|$, $|B|$, $|C|$, $|A \cap B|$, $|A \cap C|$, $|B \cap C|$, and $|A \cap B \cap C|$.

Exercise 3.7 Try to conjecture a formula that expresses $|A_1 \cup \cdots \cup A_k|$ in terms of $|A_i|$, $|A_i \cap A_j|$, $\ldots$, $|A_{i_1} \cap \cdots A_{i_{k-1}}|$ and $|A_1 \cap \cdots \cap A_k|$.

4 Applying Inclusion-Exclusion to Derangements

For $n \geq 1$, denote by A_i the set of permutations of n objects that fix object i.

Exercise 4.1 Let $n = 3$. Determine $|A_i|$ for $i = 1, 2, 3$. Next, determine $|A_i \cap A_j|$ for $1 \leq i < j \leq 3$. Finally, determine $|A_1 \cap A_2 \cap A_3|$. Use this to determine the value of $|A_1 \cup A_2 \cup A_3|$. What are the permutations not included in $A_1 \cup A_2 \cup A_3$?

Exercise 4.2 Let $n = 4$. Repeat the above exercise. Look at the value of all possible $|A_i|$ for $1 \leq i \leq 4$, $|A_i \cap A_j|$ for $1 \leq i < j \leq 4$, $|A_i \cap A_j \cap A_k|$ for $1 \leq i < j < k \leq 4$, and $|A_1 \cap A_2 \cap A_3 \cap A_4|$. Which permutations are not included in $A_1 \cup A_2 \cup A_3 \cup A_4$? How many are there?

Exercise 4.3 For general $n > 0$, what is the size of $|A_i|$? What is the size of $|A_i \cap A_j|$ for $i \neq j$? What is the size of $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ where $1 \leq k \leq n$?

Exercise 4.4 For general n, how many pairs (i, j) with $1 \leq i < j \leq n$ are there? How many triples (i, j, k) with $1 \leq i < j < k \leq n$ are there? How many $k-$tuples $(i_1, i_2, \ldots, i_k)$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ are there?

Exercise 4.5 Using the previous two exercises, and your knowledge of the generalized inclusion-exclusion formula, try to conjecture what the value of $|A_1 \cup \cdots \cup A_n|$ is. This counts the number of permutations that fix at least one object. Subtracting this from the total number of permutations will give the number of derangements on n objects.

5 Recursive formula

A *sequence* of real numbers is a set of real numbers indexed by a subset of the integers. In practice, it is generally indexed by the nonnegative integers.

Examples of sequences are as follows:

1. $1, 1, 1, 1, 1, 1, \ldots$

2. $1, 2, 4, 8, 16, \ldots$

3. $1, 1, 2, 3, 5, 8, \ldots$

4. $-1, 1, -1, 1, -1, \ldots$

In many cases, a formula for the sequence may be hard to write down but relating certain terms with previous terms can be an easier task. When one can write down a formula expressing an element in the sequence in terms of previous elements, and applying this formula consistently from a certain index, we call this a *recursive definition*. This requires setting some initial conditions to make the formula make sense.

In the first example above, setting $a_0 = 1$ as the initial condition, one can the express the sequence as $a_{i+1} = a_i$ for $i \geq 0$.

In the second example, setting $a_0 = 1$ as the initial condition, one can express subsequent terms as $a_{i+1} = 2a_i$ for $i \geq 0$.

In the third example, we require two terms to be defined for the initial conditions. Setting $a_0 = a_1 = 1$, the sequence then follows the recursive definition $a_{i+2} = a_i + a_{i+1}$ for $i \geq 0$.

In the fourth example, setting $a_0 = -1$, the recursive definition is then $a_{i+1} = -a_i$ for $i \geq 0$.

Exercise 5.1 Consider the sequence $1, 3, 7, 15, 31, \ldots$. Set $a_0 = 1$. Try to express each terms a_{i+1} as a multiple of the previous term plus a constant term, i.e. $a_{i+1} = C \cdot a_i + D$ for constants C, D.

Exercise 5.2 Consider the sequence $1, 2, 5, 26, 677, \ldots$. Try relating the square of each previous term with the next term to determine a recursive relation.

6 Applying recursion to derangements

We now apply the notion of recursive sequences to derangements. Set $\mathcal{D}_n$ to be the set of derangements in permutations on n objects and set $D_n = |\mathcal{D}_n|$. We will try to write the D_n in terms or D_{n-1} and D_{n-2}. to this end, define $\mathcal{R}_k$ to be the set of derangements where k is in the n^{th} position and set $r_k = |\mathcal{R}_k|$.

Exercise 6.1 For each derangement, examine the n^{th} position. What value must not be in that position? Argue that $\mathcal{D}_n = \bigcup_{i=1}^{n-1} \mathcal{R}_i$.

Exercise 6.2 What is the value of r_1? How about r_2? How about r_{n-1}? And finally, how about r_n?

Exercise 6.3 Is it possible for a derangement to be an element of $\mathcal{R}_i$ and $\mathcal{R}_j$ for $i \neq j$? Express D_n in terms of r_i for $i = 1, \ldots, n-1$.

Examine $\mathcal{R}_{n-1}$ more closely. We can consider what these derangements look like. Recall that each of these place $n-1$ in the n^{th} position.

Exercise 6.4 In $\mathcal{R}_{n-1}$, if n is placed in the $n-1^{\text{st}}$ position, what positions remain to be filled? Which numbers are available? Can you relate the positions and numbers available to a particular D_k for some k?

Exercise 6.5 In $\mathcal{R}_{n-1}$, consider the derangements in which k is placed in position $n-1^{\text{st}}$ position where $k \neq n, n-1$. What happens if the position in which the n^{th} object is placed is replaced by the $n-1^{\text{st}}$ object, (for $n = 5$, if the objects $1,\ 2,\ 3,\ 4,\ 5$ are in positions $(3,5,1,2,4)$ then the new positions are given by $(3,4,1,2)$). Is this a derangement in $\mathcal{D}_{n-1}$? Can one go backwards from a derangement in $\mathcal{D}_{n-1}$ to a derangement in $\mathcal{R}_{n-1}$ in which n is not in the $n-1^{\text{st}}$ position?

In the previous two exercises, you showed that r_{n-1} can be written as the sum of two different D_k values.

Exercise 6.6 Try to use the previous exercises to write D_n in terms of multiples of different D_i terms for $i < n$.

From this expression, we can write a second more compact recurrence relation.

Exercise 6.7 Using this expression, show that $D_n = nD_{n-1} + (-1)^n$ for $n \geq 2$.