Curvature, Angle Defect, and the Gauss-Bonnet Theorem

Chapel Hill Math Circle
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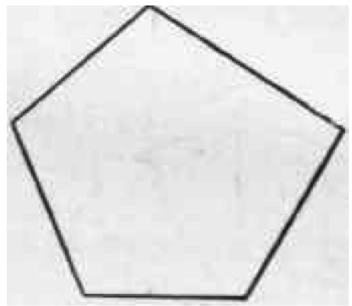
1 Warm-up

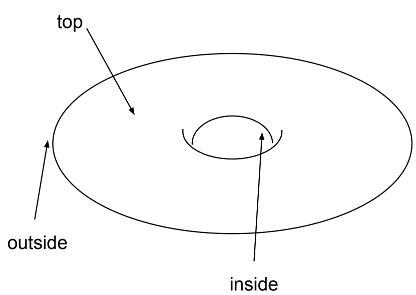
What is the sum of the interior angles of a convex polygon (in the plane) with n sides? Prove it!



2 Introduction

A piece of paper is flat. The surface of a watermelon is curved. Our goal is to define and quantify curvature to discover patterns and relationships. Curvature is a major topic in the field of Differential Geometry, and has some surprising applications.

1. Which would you say is more curved, a piece of the surface of an orange or the same size piece of the surface of a watermelon?
2. Examine the surface of a bagel. How does the curvature on the inside of the hole compare to the curvature on the outside or on the top?

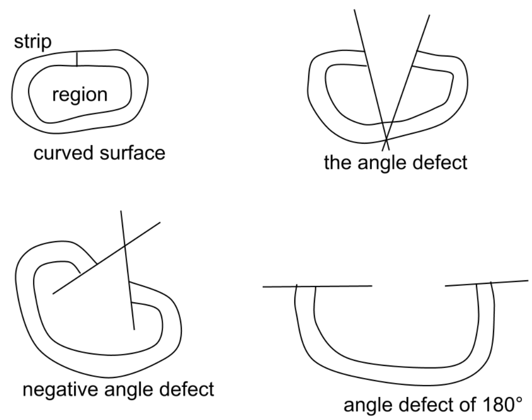


3 Angle Defect

If you take a piece of the skin of a sphere, you can't flatten it onto a plane without either stretching it or tearing it. If you've spent much time flattening orange peels, then you probably already know this. Even a small piece needs to be ripped to flatten it on the table.

One way to measure the curvature of a region of a surface is to cut a narrow ring from the boundary of the region, cut the ring open into a strip and flatten this strip onto the table, so that it opens up. The curvature of the region of surface that was bounded by the strip is the angle by which the strip opens up, which is also called the *angle defect*. Angles can be measured in degrees or in radians (360° equals 2π radians).

If the strip meets up with itself perfectly, then the region has zero angle defect, i.e. zero curvature. Sometimes, the strip doesn't meet up because it doesn't curl enough. This is positive curvature. Sometimes the strip doesn't meet up because it curls around too much and overshoots. This is negative curvature.



Note that the region must have the "topology of a disk" for the angle defect definition to work. In other words, the region should not contain any holes or handles. So a small piece of the surface of a bagel is fine, but a large piece that contains the entire hole in the middle is not okay.

3. Estimate the curvature of some of these vegetables and fruits by cutting and flattening strips that surround small regions.

   - orange peel
   - banana peel
   - cabbage
   - kale

   You'll need to pay attention not only to the angle, but also to how the strip curls around, keeping in mind that zero curvature is a strip that comes around and meets itself. Be careful about 180°'s and 360°'s (or about π's and 2π's).

4. Calculate the curvature of the upper hemisphere of a round sphere.

5. What is the curvature of a region of a flat piece of paper?

6. What is the curvature of a piece of the surface of a cylinder?

   Note: One nice feature about using angle defect to measure curvature is that it is possible to measure curvature even at regions of a surface that are not smooth, like the cone points on a cone or vertices on a polyhedron.

7. Make a cone and measure the curvature of a small region of the cone that contains the tip of the cone, called the "cone point". Does it depend on the shape of the cone?

8. On a cube, what is the curvature of a region containing one vertex?

9. Construct a surface from equilateral triangles by putting seven triangles around each vertex. What is the curvature of a piece of this surface containing one vertex?

4 Total curvature

We can calculate total curvature of an entire surface by adding up the regional curvature of a bunch of small regions that cover the surface.

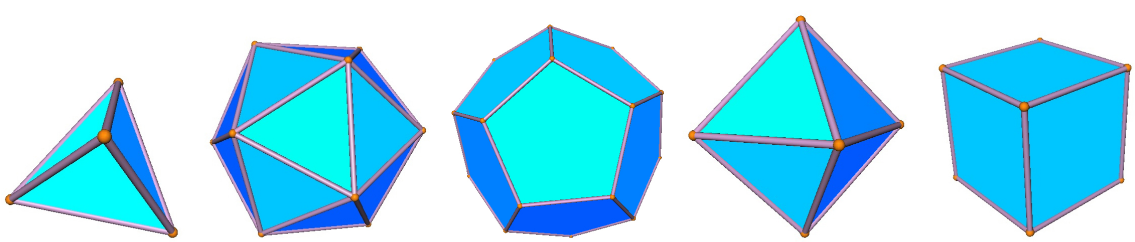
10. What is the total curvature of a sphere?

11. Which has more total curvature, the surface of an orange of the surface of a watermelon?

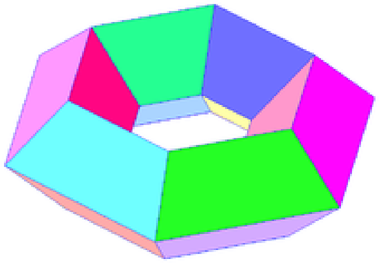
    For a polyhdedron with flat sides, like a cube or a dodecahedron, the curvature of any region that doesn't contain a vertex is 0, so we really only need to add up the angle defect around each vertex. An easy way to do this is to add up the angles at the corners of the faces that meet at the vertex and subtract from 360°. For instance, at any vertex of a tetrahedron there are three angles of 60°, so the angle defect is $360° - 3 \cdot 60° = 180°$.

12. Calculate the total curvature of the polyhedra listed below by adding up the angle defect around each vertex to get the "total angle defect".

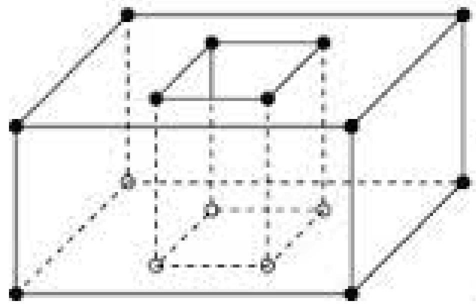
(Semi) regular polyhedron	# of faces (F) (F)	# of edges (E) (E)	# of vertices (V) (V)	Angle defect of each vertex	Total angle defect
Triangular prism					
Tetrahedron					
Cube	6	12	8	90° or $\frac{\pi}{2}$	720° or 4π
Octahedron					
Icosohedron					
Dodecahedron					
Soccer ball					
Hexagonal toroid					
Square toroid					
Multishape toroid					



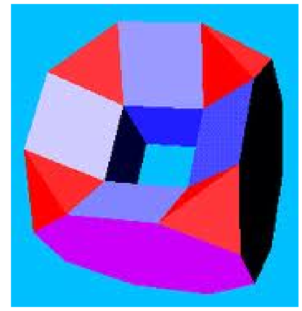
Tetrahedron Icosahedron Dodecahedron Octahedron Cube



“Hexagonal toroid”



“Square toroid”



“Multishape toroid”

The total angle defect is intimately connected with another number from topology: the Euler number. The Euler number χ is defined as $V - E + F$, where V is the number of vertices, E is the number of edges, and F is the number of faces. What is the relationship between total angle defect and Euler number? (This relationship is called Descartes Angle Defect Theorem.)

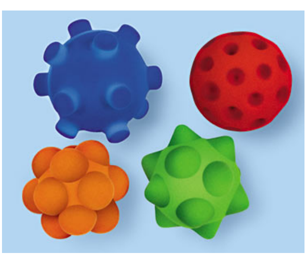
5 The Gauss Bonnet Theorem

The Gauss Bonnet Theorem generalizes Descartes Angle Defect Formula to surfaces that are not polyhedra. It says that the total curvature of any closed surface is $2\pi\chi$ (or $360°\chi$, if you are using degrees instead of radians), where χ is the Euler number of the surface.

The Gauss Bonnet Theorem is amazing because it relates curvature (geometry) to Euler number (which depends only on topology). It tells us that even if earthquakes form new mountains on Earth, creating additional positive curvature, that new positive curvature has to be exactly balanced by new saddle points, with negative curvature, so that the total curvature remains unchanged.

13. If a new hill creates positive curvature on a previously flat plane, where is the compensating negative curvature?

14. What is the total curvature of each of the surfaces shown below?

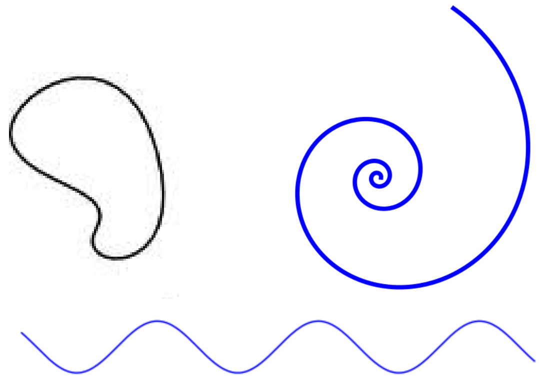


15. So far, we have only considered surfaces without boundary. State and prove a version of Descartes Angle Defect Formula for surfaces with boundary. Hint: be careful about how you define angle defect for vertices on the boundary.

6 Curvature of Curves in the Plane

Before we give an alternative version of curvature of a surface, let's drop down a dimension and think about curvature of 1-dimensional curves.

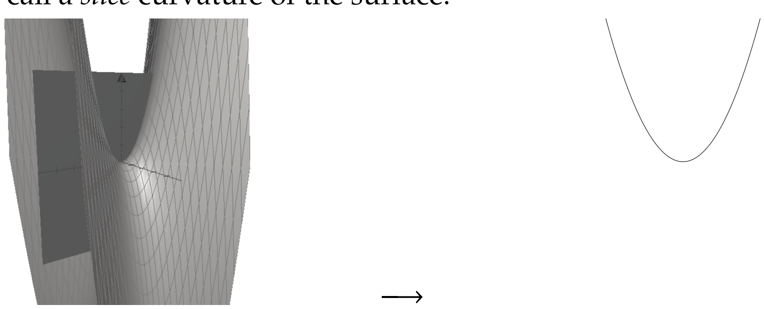
16. For each of the three curves drawn below, describe the curvature at various points along the curve. For each curve, where is the curvature biggest and where is it smallest?



17. What is the curvature at a point of a straight line?
18. Draw another curve (not a straight line) that has the same curvature at all of its points. How could you quantify its curvature as a number?
19. Extend this idea to define the curvature at points of other curves, like the spiral or the sine curve (the wave above).

7 Gaussian Curvature at a Point

Once we have a notion of curvature of curves in a plane, we can use this to define the curvature at a point on a surface in various directions. Take a point on the surface, and imagine slicing the surface by a plane perpendicular to the surface at that point. The intersection of the 2-dimensional surface and the plane makes a 1-dimensional curve in the plane. This 1-dimensional curve has a curvature that we'll call a *slice* curvature of the surface.



At any point, there could be infinitely many slice curvatures, depending on which direction you slice the surface with a plane. If the surface bends in opposite directions, as happens at a saddle point, then we consider some slice curvatures positive and others negative.

At any point of the surface, the maximum and minimum slice curvatures are called the *principal curvatures* at that point. It is a fact (that we won't prove) that the principal curvatures always lie in perpendicular directions.

One way to get a single number for the curvature of a point on the surface is to multiply together the principal curvatures. This product is called the *Gaussian curvature* at that point on the surface.

20. According to this definition, is the curvature at a point on a watermelon bigger or smaller than the curvature at a point on an orange?

21. Estimate the curvature at various points on a bagel. Where is the curvature greatest and least? Where is the curvature positive, negative, and zero?

22. What is the curvature of a point on a cylinder?

23. What is the curvature of a point on the side of a cone?

24. What are the units of curvature at a point?

25. Calculate the curvature at each point of a sphere of radius 9 cm.

26. Calculate the regional curvature of the upper hemisphere of a sphere of radius 9 cm. Since the curvature at each point is the same, you can find the regional curvature by multiplying the curvature at a point by the area of the region.

27. What are the units of regional curvature?

28. How do your calculations of curvature in this section compare to you calculations using the angel defect definition of curvature?

8 Reference

1. Much of this material is taken from *Geometry and the Imagination Course Notes* by John Conway, Peter Doyle, Jane Gilman, and Bill Thurston. See http://www.geom.uiuc.edu/docs/doyle/mpls/handouts/handouts.html