Problems about Integers

January 27, 2018

1 Vocabulary

1. A natural number is a counting number: $1, 2, 3, 4, \cdots$.
2. A divisor is a natural number that divides another number. For example, 4 is a divisor of 12, but 10 is not. What are all the divisors of 12?
3. 5! is pronounced "five factorial" and means $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ all multiplied together. Similarly, $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, etc.

2 Divisibility

1. How many divisors does the number 33 have? The number 66? The number 99?
2. Given two different prime numbers p and q, find the number of different divisors of the numbers
   (a) pq
   (b) p^2q
   (c) p^2q^2
   (d) p^nq^n
3. Prove that the product of any three consecutive natural numbers is divisible by 6.
4. Prove that the product of any five consecutive natural numbers is
   (a) divisible by 30
   (b) divisible by 120
5. Find the smallest natural number n such that $n!$ is divisible by 990
6. For some number n, can the number $n!$ have exactly five zeros at the end of its decimal representation?
7. Prove that if a number has an odd number of divisors, then it is a perfect square.
8. Tom multiplied two two-digit numbers on the blackboard. Then he changed all the digits to letters (different digits were changed to different letters and equal digits were changed to the same letter). He got $AB \cdot CD = EEFF$. Prove that Tom made a mistake somewhere.
9. Can a number written with one hundred 0's, one hundred 1's, and one hundred 2's be a perfect square? Hint: think about divisibility by 3 and by 9.
10. The numbers a and b satisfy the equation $56a = 65b$. Prove that $a + b$ is composite.[1]

[1] This week's problems are from *Mathematical Circles: The Russian Experience*