CHMC Advanced Group: Protecting an art gallery

01/27/2018

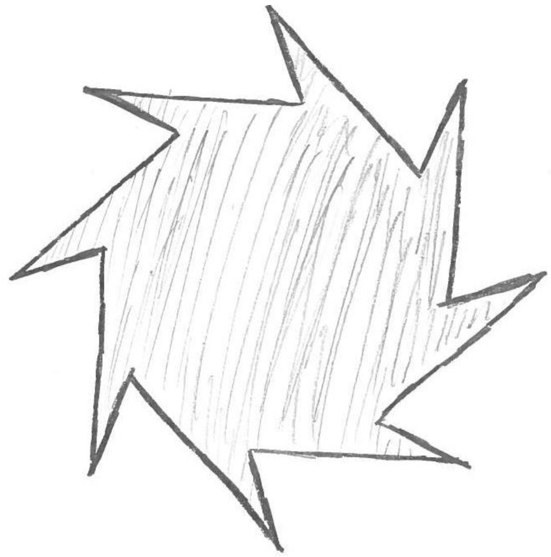
Suppose you own an art gallery that has 1) a single room, 2) walls that are straight lines, and 3) some very expensive pieces of art. You want to place some guards around the gallery so that no portion of the gallery is unguarded, but guards are expensive and you don't want to hire more than you need. What is the smallest number of guards you need to protect your gallery? What if you don't know the shape of the gallery beforehand, but you do know it will have w walls (maybe it's still being constructed)? In this case what is the smallest number of guards needed, as a function of w, to ensure your gallery is protected?[1]

This worksheet will explore these questions. The first section explores the first question posed, where we know the layout of the gallery. The second section generalizes this question to the situation where we don't know the layout of the gallery, but just the number of walls. The third section explores a few special families of art galleries that are important for one reason or another, and the next two sections, one of which is optional, outline the solution to this problem. The very last section has some optional extra exercises.

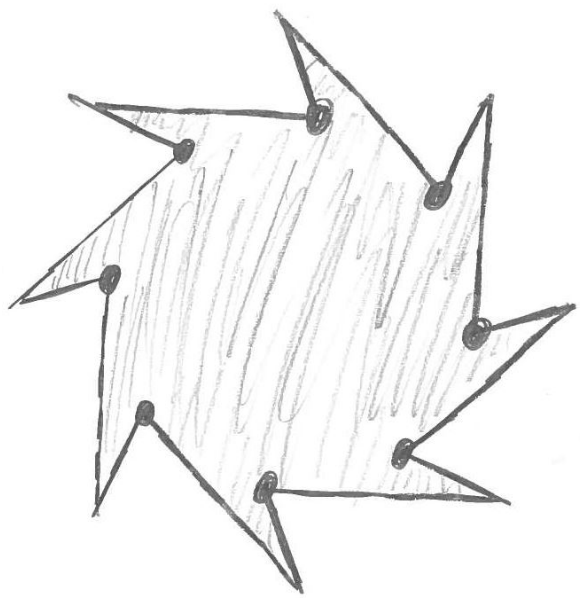
1 The problem

Consider the art gallery below, called the **sunflower gallery**.

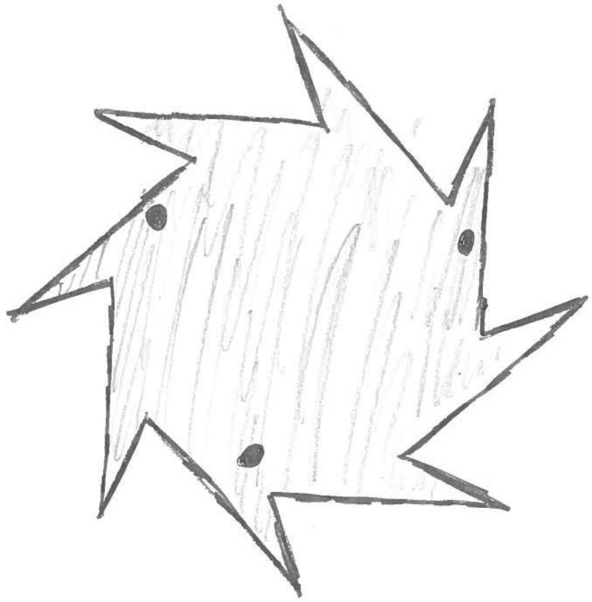
[1] This worksheet is heavily adapted from the highly recommended Chapter 3 of "How to Guard an Art Gallery" by T.S. Michael.



Since we're trying to place enough guards so that no portion of this gallery is left unwatched, we could start by placing guards at each of the inward pointing corners, like so:



While this gallery is now protected, it should be clear that some of the guards are unnecessary. Also, we aren't constrained to just placing guards along the walls or at corners. For example, here is a valid guard placement (with fewer guards) that protects the gallery:



Exercise 1.1 Verify that in the last figure, every portion of the gallery is visible to at least one guard, so that the gallery is in fact protected.

Could we still guard this gallery with fewer guards?

Exercise 1.2 No; prove this. Hint: what is the greatest number of the small wedge-shaped rooms one guard can watch?

Thus we've shown that for the sun gallery 3 guards suffice, and we can't guard the gallery with fewer than 3 guards. This number we will call the **guard number** for this specific gallery.

In general, for a specific gallery G we are interested in the question: *what is the minimum number of guards required to protect the gallery G?* The guard number for a gallery G will be denoted guard(G). For example, guard(sunflower gallery) = 3.

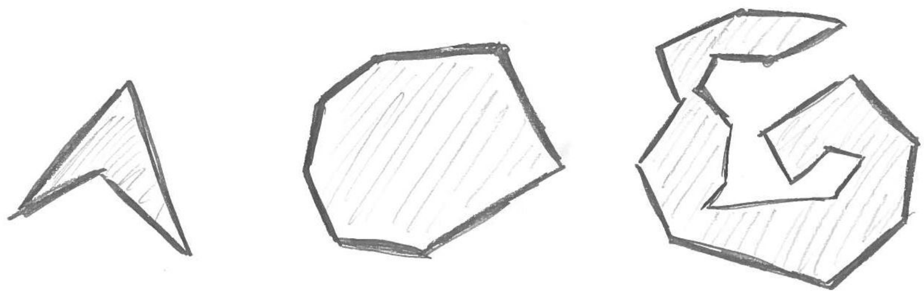
Note that, to prove guard(G) $= n$ for some gallery G, we need to show two things:

1. G can be protected by n guards.

2. G cannot be protected by fewer than n guards.

For example, we can easily guard the sunflower gallery by placing a guard at every corner, for a total of 16 guards. But we can guard G with fewer than 16 guards, showing guard(sunflower gallery) ≤ 16. In fact we can guard the sunflower gallery with 3 guards, which establishes guard(sunflower gallery) ≤ 3, i.e. the first fact we needed to show. For the second fact, we argued that this gallery needs at least 3 guards, so guard(sunflower gallery) $\geq$ 3. Putting these two inequalities together proves that guard(sunflower gallery) = 3.

Exercise 1.3 What is guard(G) for the galleries G below?



Exercise 1.4 Recall that a polygonal region G (like the art galleries we've been considering) is **convex** if for any two points x, y in G, the line segment connecting x to y is also in G. Which of the galleries G from the last exercise are convex?

Exercise 1.5 Show that guard$(G) = 1$ if G is convex.

Exercise 1.6 If guard$(G) = 1$, does G need to be convex?

2 The problem generalized

In general computing guard(G) for some gallery G is a hard problem. For example, computing guard(G) for the gallery G below, as well as their placements, seems like a daunting task. Fortunately if we generalize the problem slightly, we'll see that we can get a nice upper bound on the number of guards required, as well as get a procedural method for placing the guards.

Let

$g(w) =$ the maximum number of guards required among *all* galleries with w walls.

Another way to think of this function is that $g(w)$ is the maximum of guard(G), as G ranges over all possible w-walled galleries, i.e.

$$g(w) = \max\{\text{guard}(G_w) \colon G_w \text{ is a } w\text{-walled gallery}\}.$$

Then the generalized art gallery problem is the following: *what is the value of $g(w)$ for $w = 1, 2, \ldots$?*

To get a handle on this question, as well as the function $g(w)$, let's work out some specific cases. Note that to show $g(w) = n$ for some integer n, we need to establish two facts (similar to how we could establish guard$(G) = n$ in the last section):

1. Every w-walled gallery can be protected by n guards.

2. There is a w-walled gallery that cannot be protected by fewer than n guards.

For example, the sunflower gallery has 16 walls and can be protected by 3 guards, but cannot be protected by 2 guards. This example shows that $g(16) \geq 3$. If we wanted to prove that $g(16) = 3$, we would need to show that there is not some gallery $G_1 6$ that requires at least 4 guards to be completely guarded. It turns out that $g(16) = 5$, which we will prove later.

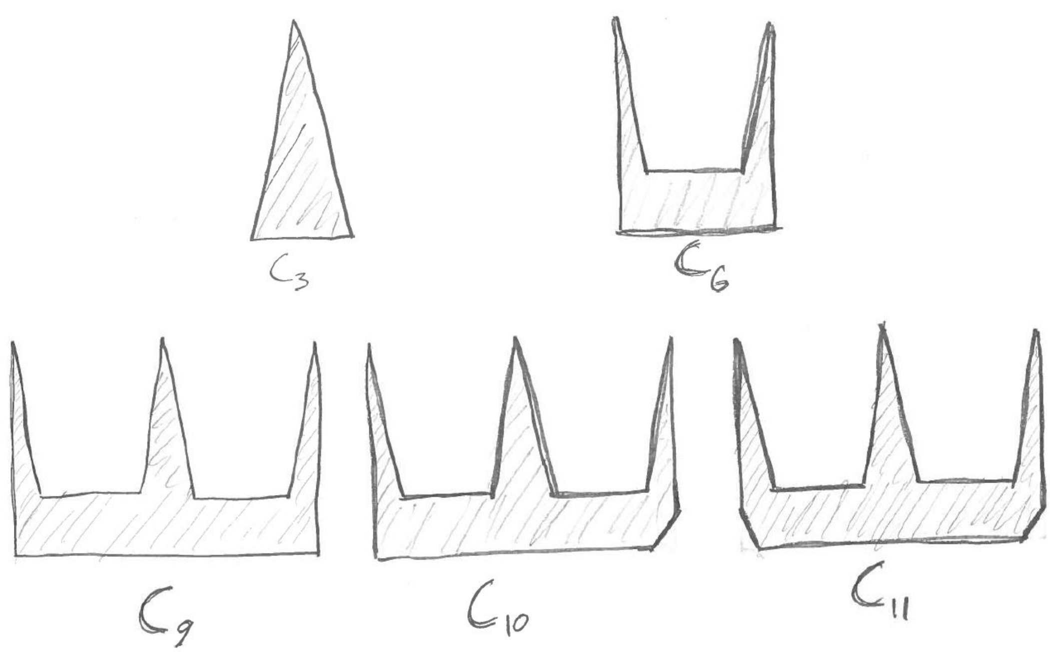
Exercise 2.1 What does the number $g(3)$ represent? Show that $g(3) = 1$.

Exercise 2.2 What does the number $g(4)$ represent? Show that $g(4) = 1$. Note: you'll need to consider two cases, first when G_4 is a convex quadrilateral, and second when G_4 is not convex. Draw some examples of 4-walled galleries to get a sense of what happens.

Exercise 2.3 *Show that $g(5) = 1$. You can draw specific 5-walled galleries to get a feel for what happens, but your argument should hold for *any* 5-walled gallery.

3 Crown galleries

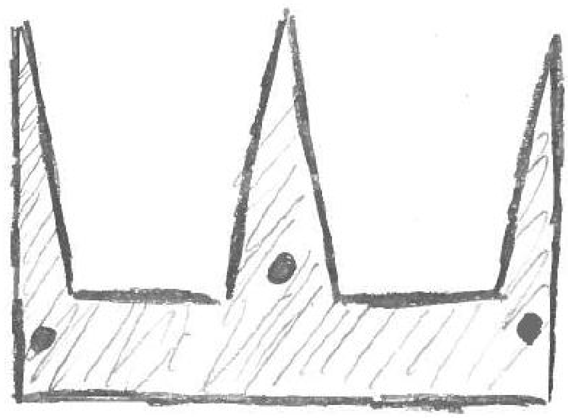
An important class of galleries are the crown galleries C_k, where k is the number of walls. In practice we think of these galleries as either being of the form C_{3k}, C_{3k+1}, or C_{3k+2}. It is easier to define the shape of these galleries by a series of examples:



Thus, the gallery C_{3k} has k "spires", and the +1 or +2 denote the number of dents in the crown. Note that, when you draw a crown gallery, the spires should be sufficiently pointy and sufficiently far apart (as in the examples).

Exercise 3.1 Draw C_{16} and C_7.

As an example, C_9 is drawn below, with a placement of guards.



Show that guard(C_9) = 3. I.e., why can't we guard C_9 with fewer guards?

Exercise 3.2 What are the numbers guard(C_{16}) and guard(C_7)?

Exercise 3.3 In general, what is guard(C_w) for any w? You may need to consider three different cases.

4 The solution

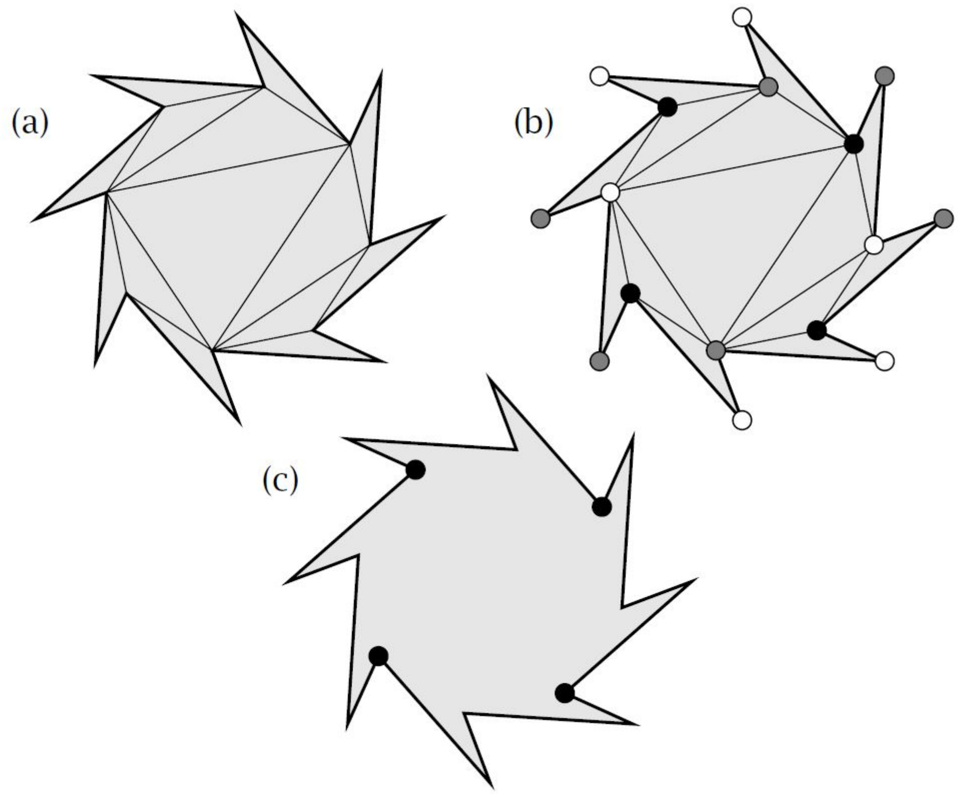
The surprising answer is that $g(w) = \lfloor w/3 \rfloor$, i.e. *any* gallery with w walls can be guarded by no more than $\lfloor w/3 \rfloor$ guards. Again, there may be galleries that can get by with fewer guards, but $\lfloor w/3 \rfloor$ can successfully guard every inch of the gallery irregardless the shape of the gallery. Here $\lfloor x \rfloor$ is the greatest integer $\leq x$. For example, $\lfloor 2.319 \rfloor = 2$ and $\lfloor \pi \rfloor = 3$. The rest of this worksheet is aimed at proving this value for $g(w)$.

There are two main steps in establishing this number:

1. Triangulate the gallery by connecting vertices of the original gallery with straight lines. These connections are not allowed to cross one another.

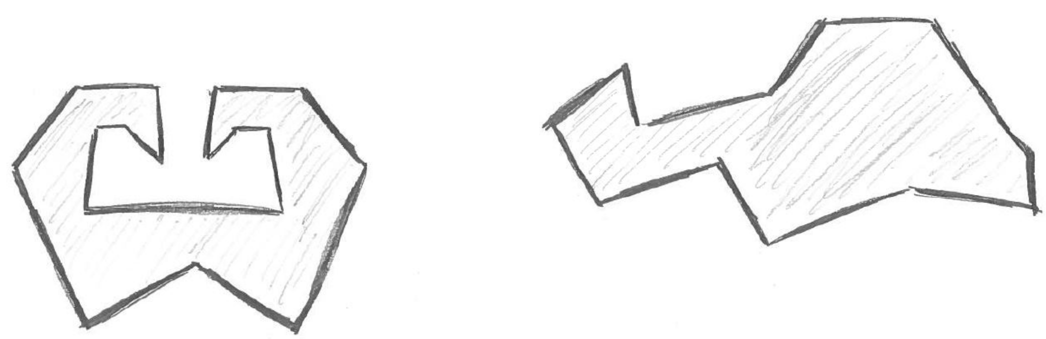
2. Color the vertices of the triangulation with 3 colors so that no two vertices of a triangle have the same color.

After we triangulate and color the gallery, we can pick the color that appears the fewest number of times. If we place a guard at each of these vertices, we will have succeeded in guarding the gallery. Let's look at an example involving the sunflower gallery.



In figure (a) we have a triangulation of the gallery. Note that this isn't the only triangulation we could have used. In figure (b) the vertices of the triangulation have all been colored. You can see that there are four black vertices, 6 white vertices, and 6 grey vertices. Figure (c) then shows one possible placement of guards. Note that, in this specific case, we have a solution with fewer guards, but the procedure itself works for any shape of gallery.

Exercise 4.1 What is a triangulation for each of the art galleries in the figure below?



Exercise 4.2 Using these triangulations, give a colouring to the vertices to each of the art galleries. Where could you place guards to guard each gallery?

In general, the color that appears the fewest amount of times in a w-walled gallery is $\lfloor w/3 \rfloor$, which proves our assertion.

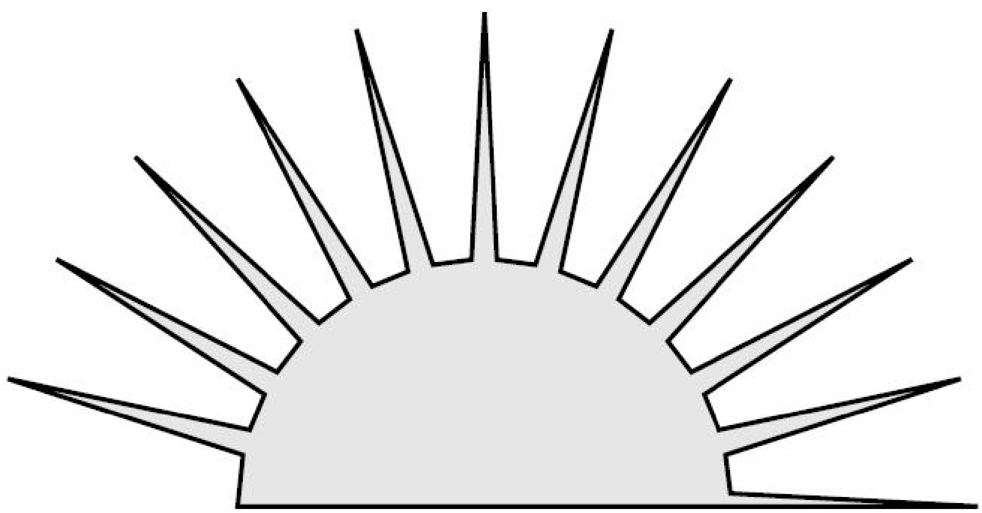
Exercise 4.3 *** Prove this, i.e. prove that in any triangulation and any coloring of a w-walled gallery, the color that appears least only occurs $\lfloor w/3 \rfloor$ times.

All that is left is to show that there are galleries we can't guard with fewer than $\lfloor w/3 \rfloor$ guards.

Exercise 4.4 Recall the number guard(C_w) for the crown galleries C_w. Why does this establish that $g(w) = \lfloor w/3 \rfloor$?

On the other side of the spectrum, this procedure can often times produce some very non-optimal results. Part of the issue is that the algorithm only places guards at corners, and not within the gallery. This is something you may have already noticed with the sunflower gallery.

The next three exercises deal with the sunrise gallery, which is an example of how far off the algorithm can be from giving an optimal solution.



Exercise 4.5 What is guard(sunrise gallery)? Where would you place the guards to guard this gallery?

Exercise 4.6 How many walls w does the sunrise gallery have? What is $\lfloor w/3 \rfloor$ in this case?

Exercise 4.7 Using the algorithm for placing guards, via triangulation and coloring vertices, where would you place guards to guard the sunrise gallery (as in the algorithm)?

One interesting question is: once a gallery has been triangulated, how many possible ways can you color the vertices? It turns out that you don't have many options.

Exercise 4.8 Prove that once one triangle is colored, the coloring of every other triangle is completely determined.

5 *Extra problems

These extra problems are taken directly from the book mentioned in the introduction.

Exercise 5.1 Construct an art gallery G such that

1. the gallery can be protected by one guard;
2. it is possible to post guards at seven corners and not protect the entire gallery.

Exercise 5.2 Construct an art gallery G such that

1. the gallery can be protected by two guards, but not by one guard;
2. it is possible to post guards at twenty-nine corners and not protect the entire gallery.