CHMC:
An application of spheres

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In this worksheet, we'll explore an interesting application of spheres. In particular, we'll see an interesting connection between spheres and planar graphs, and use this to show that if we connect any two planar graphs (in a suitable sense of "connect"), then the resulting graph is also planar.

The worksheet is organized as follows: the first section will formalize some aspects of line segments, namely define what a line segment is and learn how to work with these things in a coordinate sense. The second section will introduce the fundamental tool in the connection between a sphere and planar graphs: stereographic projection. The final section focuses on the connection between planar graphs and spheres.

1 Line segments

A line segment is, as you may remember, a straight line connecting two points. Almost all of the line segments we'll work with in this worksheet will be finite, i.e. won't go off to infinity.

The natural starting point is to look at line segments in the real line, which can be thought of as "connected portions" of the whole real line. In particular, we'll define a line segment $[a, b]$ in the real line to be the set of numbers less than b and greater than a.

Exercise 1.1 Show that the set $[a, b] = \{t \colon a \leq t \leq b\}$ can be expressed in the following two ways:

$$[a, b] = \{(b - a)t + a \colon 0 \leq t \leq 1\},$$
$$\text{and } [a, b] = \{a(1 - t) + tb \colon 0 \leq t \leq 1\}.$$

Hint: start by showing $(b - a)t + a \leq b$ for $0 \leq t \leq 1$, etc.

We define the line segment connecting two points p and q (where p and q can be 1-, 2-, or 3-dimensional points) to be the set of points

$$\{(p - q)t + q, 0 \leq t \leq 1\}.$$

The next few exercises all involve working with line segments in 2- and 3-dimensions.

Exercise 1.2 Using the definition above, show that, in the plane, the line segment connecting the points (a_1, a_2) and (b_1, b_2) is the set of points

$$\{(tb_1 + (1-t)a_1, tb_2 + (1-t)a_2), 0 \leq t \leq 1\}.$$

Exercise 1.3 What is the line segment from the point $(1, 3)$ to the point $(-2, 4)$ in the plane? Sketch the line segment.

Exercise 1.4 In the line $(p-q)t + q$, where $0 \leq t \leq 1$ and p, q are points on the real line, plane, or 3-dimensional space, what is special about the points corresponding to $t = 1/2$? to $t = 0$? to $t = 1$?

Exercise 1.5 What is a similar representation for the line segment connecting the points (a_1, a_2, a_3) and (b_1, b_2, b_3) in 3-dimensions? Prove your conjecture. Hint: go back to the definition of a line segment connecting two points.

Exercise 1.6 What is the line segment connecting the points $(1, 1, 1)$ and $(1, -1, 1)$? Sketch this line segment.

Exercise 1.7 As a special case, we let $(a_1, a_2, a_3) = (0, 0, 0)$. What is the line segment from (a_1, a_2, a_3) to (b_1, b_2, b_3) now? Sketch the line segment.

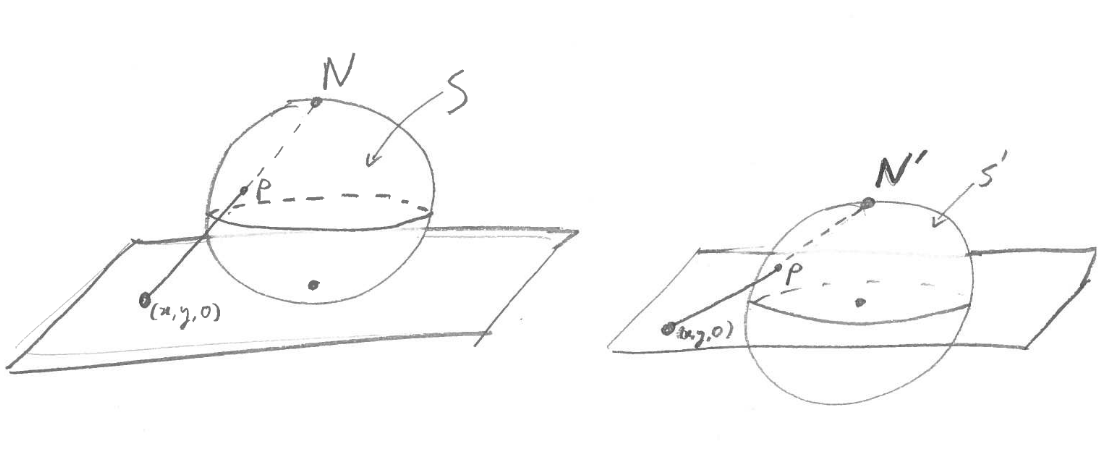
Exercise 1.8 For the line segment in the last exercise, what happens if we let t go to infinity? I.e. what happens if we let t take values past where we have defined it to be?

2 Stereographic Projection

The key idea with stereographic projection is that, to each point in the plane we associate a point on a sphere. By how we define this association though, we can go backwards and associate points on a sphere with points in the plane.

Imagine (or look at the sketch below) a sphere S of some radius, tangent to the plane at the origin of the plane. If we turn on a lightbulb at the north pole N, rays of light will extend through the sphere and eventually touch the plane.

Suppose this ray of light, this straight line segment, passes through the sphere at the point p, and meets the plane at a point $(x, y, 0)$. Stereographic projection associates the point $(x, y, 0)$ to the point p.



For now, use this analogy of stereographic projection and light rays to answer the next two exercises.

Exercise 2.1 Where does the north pole of the sphere get sent under stereographic projection?

Suppose that instead of the sphere being tangent to the plane at the origin, the sphere is centered at the origin. Call this new sphere S' (see the above sketch). We can still define a stereographic projection from the plane to S' by again considering straight line segments (or rays of light) emanating from the north pole N'.

Exercise 2.2 Where do points in the upper hemisphere get sent for the second sphere S'? What about points in the lower hemisphere?

Let's coordinatize the sphere S we're using for stereographic projection. Recall that, in 3-dimensions, a sphere of radius r centered at the point (a_1, a_2, a_3) can be expressed as the points (x, y, z) that satisfy

$$(x - a_1)^2 + (y - a_2)^2 + (z - a_3)^2 = r^2.$$

The sphere S we'll work with will have radius $r = 1$, and will be centered at the point $(0, 0, 1)$.

Exercise 2.3 With this coordinatization, at what point (a, b, c) is the north pole N of S?

Exercise 2.4 What is the equation of the sphere S?

Exercise 2.5 Let N be the north pole of S, and let $(x, y, 0)$ be a point in the plane. What is the line segment connecting N and $(x, y, 0)$?

Exercise 2.6 Stereographic projection sends the point $(x, y, 0)$ to a point p on S, where p is the point of intersection between S and the line segment connecting N and $(x, y, 0)$. If we have this line represented as $((x, y, 0) - N)t + N$, for what value of t do we get the point p? Hint: express the line segment connecting N and $(x, y, 0)$ in terms of its coordinates, plug those coordinates into the equation of the sphere S, and solve for t.

Exercise 2.7 Using results from the last exercise, show that the point p has coordinates

$$p = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{2(x^2+y^2-1)}{1+x^2+y^2} \right).$$

These coordinates for p are the result of applying stereographic projection to the point $(x, y, 0)$. In other words, stereographic projection takes the form of a function F from the plane to the sphere, where

$$F(x, y, 0) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{2(x^2+y^2-1)}{1+x^2+y^2} \right).$$

Exercise 2.8 ** Show that stereographic projection is one-to-one. In other words, show that if two points $(a_1, b_1, 0)$ and $(a_2, b_2, 0)$ get sent to the same point p, then we must have $a_1 = a_2$ and $b_1 = b_2$.

Exercise 2.9 What happens as we let the point $(x, y, 0)$ head off to infinity, say by letting $x \to \infty$ and keeping y fixed? Does this agree with previous observations?

Exercise 2.10 (Requires some knowledge of complex numbers) Consider the point $(x, y, 0)$, and let $z = x + iy$. Recall that we have the complex conjugate $\bar{z} = x - iy$. Express the coordinates of the point p, the stereographic projection of $(x, y, 0)$, in terms of z and $\bar{z}$.

3 Planar graphs

A **graph** is a collection of points $\{v_1, ..., v_n\}$, called vertices, and a collection of edges connecting vertices. We think of/visualize these edges as curves (not necessarily straight lines) connecting two vertices.

A graph is **planar** if the graph can be drawn in the plane. This means that we can place the vertices in the plane, and connect them with edges, such that no two edges intersect. There are plenty of examples of non-planar graphs, but in practice proving that a graph is non-planar can be fairly difficult. We won't need any of that for this worksheet though.

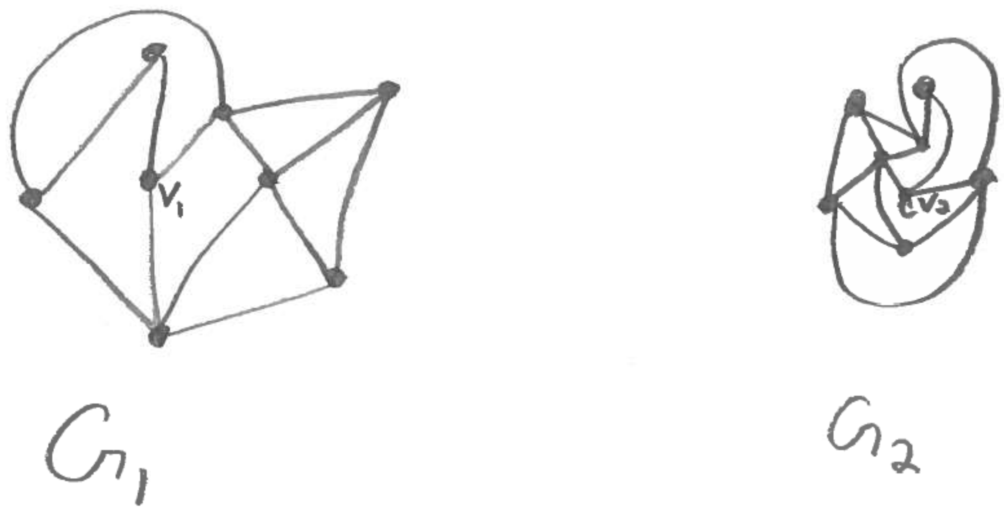
The key connection between planar graphs and spheres is the following:

Exercise 3.1 Using the stereographic projection (or otherwise), prove that a graph is planar if and only if the graph can be drawn on a sphere. Hint: don't use formulas, think about the stereographic projection via pictures.

Thus, being able to draw a graph on a sphere means you can draw it in the plane, and vice versa. Sometimes drawing a graph on a sphere is much easier to do.

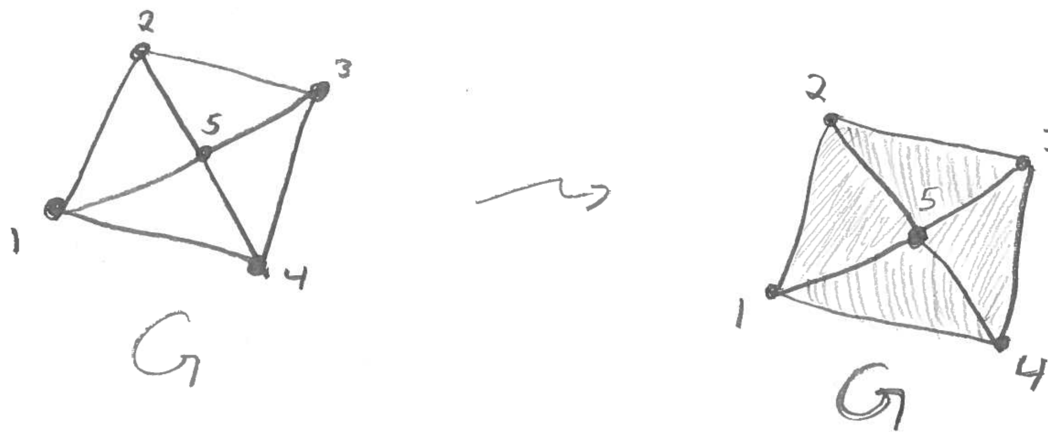
Now suppose we have two planar graphs G_1, G_2, and take a vertex v_1 in G_1, and a vertex v_2 in G_2. The problem we'll be interested for the remainder of this worksheet is if the graph we get by connecting v_1 to v_2 is planar or not; turns out it always will be!

Exercise 3.2 In the graph below, connect vertices v_1 in G_1 and v_2 in G_2 (v_2 has an arrow pointing to the actual point). Redraw the resulting graph to show that it is planar.



For the next three exercises, we'll need the notion of the **boundary** and **interior** of a graph. We'll define the boundary via a construction: take a graph G and shade in any bounded region, i.e. shade everything in except for the unbounded region of the plane. Any vertex that touches a shaded portion *and* an unshaded portion of the plane is in the boundary of the graph; every other vertex of G is in the interior of the graph.

In the example below, vertices $1, 2, 3,$ and 4 are boundary vertices, whereas vertex 5 is an interior vertex.



Exercise 3.3 Draw your own planar graph! What are its boundary and interior vertices?

Exercise 3.4 Show that if G is a planar graph, and v is in the interior of G, then we can redraw G in the plane so that v is on the boundary of G. Hint: use spheres!

Exercise 3.5 Using the last exercise (or otherwise), prove that if G_1 and G_2 are planar graphs, and v_1 is any vertex of G_1, v_2 is any vertex of G_2, then the planar graph you get by connecting v_1 and v_2 with an edge is planar.

Neat!