Chapel Hill Math Circle: Symmetry and Fractals

10/7/17

1 Introduction

This worksheet will explore symmetry. To mathematicians, a symmetry of an object is, roughly speaking, a transformation that does not affect the appearance of the object.

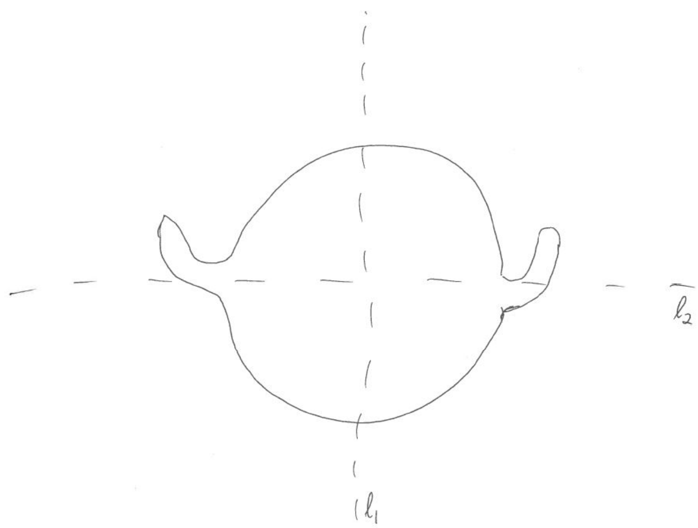
The second section explores four specific types of symmetries: reflectional, rotational, translational, and self-similarity. The first three are "classical," whereas self-similarity is a more modern topic of interest. Self-similarity usually manifests itself as fractals; the last section explores properties of some of the more popular fractals.

2 Symmetry

Reflectional Symmetry

A plane figure has *reflectional symmetry* if, as the name suggests, there is some reflection that preserves the shape of the figure.

For example, the object in the (badly drawn) picture is reflectionally symmetric because reflecting across the line l_1 results in the same shape, but reflecting across the line l_2 does not result in the same shape.



It is also not too hard to see that this is the only axis (for this specific figure) that gives us a symmetry. Because of this we say this shape only has one reflectional symmetry, i.e. there is only one transformation of reflection that preserves this objects shape.

To make this a little more rigorous, we say that a point p is reflected across the line l by being sent to the point p' that has the same distance to l as p does, and such that the line pp' is perpendicular to l.

Exercise 2.1 How many reflectional symmetries does a triangle have? What are they?

Exercise 2.2 How many reflectional symmetries does a square have? What are they?

Exercise 2.3 How many reflectional symmetries does a regular n-gon have? You may need to break this into 2 cases.

Note that we specified a reflectional symmetry by specifying a line l that we reflected across. The more lines we can reflect across, the more (reflectionally) symmetric the object is.

Exercise 2.4 What plane figure has the most symmetries?

Notice that in the symmetries above, all of the lines we reflected over intersected. In the case of a regular n-gon, all of the lines of reflection intersected at the same point.

Exercise 2.5 Is it possible for a plane figure to have axes of reflection that do not intersect? Why or why not?

Reflectional symmetries also make sense for 3-dimensional solids. Here, instead of reflecting across a line in the plane, we need to reflect across a plane in 3-dimensional space. A point p is reflected across this plane Π by being sent to the point p' that has the same distance to Π as p does, and such that the line pp' is perpendicular to Π.

Exercise 2.6 How many reflectional symmetries does a tetrahedron have? What about a cube?

Exercise 2.7 How many reflectional symmetries does a cylinder with a regular base have?

Exercise 2.8 With these observations in mind, what is the 3-d solid with the most reflectional symmetries?

Notice that reflections still made sense in 3-dimensions; instead of reflecting a 2-dimensional figure across a 1-dimensional object (line), we reflected a 3-dimensional figure across a 2-dimensional object (plane).

Exercise 2.9 How do you think you could reflect a 4-dimensional solid? What objects do you need?

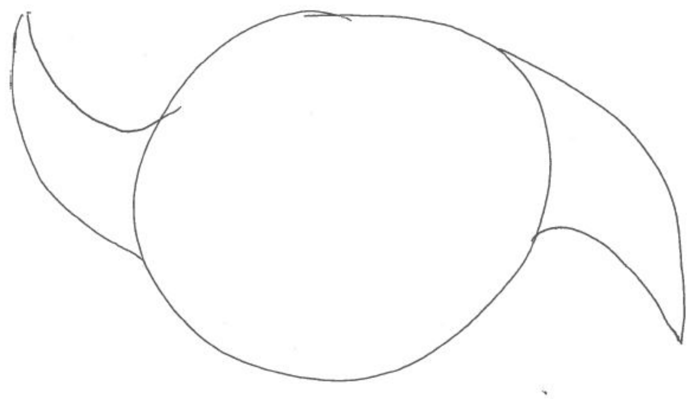
Exercise 2.10 How do you think you could reflect an n-dimensional solid?

Rotational Symmetry

The next kind of symmetry we'll explore is rotational symmetry. We say a shape has *rotational symmetry* if rotating the shape does not change the appearance of the shape.

For example, an equilateral triangle has 2 non-trivial rotational symmetries: rotating by 60 degrees to the left or right does not change the appearance of the triangle. Some people consider the rotation by 0 degrees (no rotation) as a rotational symmetry as well, but for our purposes we don't need to worry about that.

As another example, the (poorly drawn) picture below has an example of an object with just one rotational symmetry:



Exercise 2.11 What is the rotational symmetry? I.e. how many degrees (or radians) does the rotational symmetry correspond to?

Exercise 2.12 How many rotational symmetries does a square have?

Exercise 2.13 How many rotational symmetries does a regular n-gon have?

In fact, it's not too hard to come up with other examples of shapes that have some combination of rotational symmetry, reflectional symmetry, or neither.

Exercise 2.14 What is a plane figure that has reflectional symmetry, but not rotational symmetry?

Exercise 2.15 What is a plane figure that has rotational symmetry, but not reflectional symmetry?

Exercise 2.16 What is a plane figure that has neither reflectional nor rotational symmetry?

As you may have guessed, we can extend the notion of rotational symmetry to 3-dimensions. This time instead of rotating our solid around a point, we need to rotate it about a line.

Exercise 2.17 How many rotational symmetries does a tetrahedron have? If you're stuck, recall that a rotational symmetry must send vertices of the solid to vertices of the solid.

The previous exercise suggests two-ways to think of rotational symmetries: find an axis to rotate around and check that your shape looks the same after rotation, or find a rotation that sends vertices to vertices (and hence determines an axis of rotation).

Exercise 2.18 How many rotational symmetries does a cube have?

Exercise 2.19 How many rotational symmetries does a pyramid with a regular n-gon base have?

For reflectional symmetry, we saw that we could generalize to reflections of n-dimensional solids by looking at $n-1$-dimensional planes in n-dimensions. For a rotation in the plane, we had to pick a point (0-dimensional) to rotate about, and for rotations in 3-space we had to pick a line (1-dimensional) to rotate about.

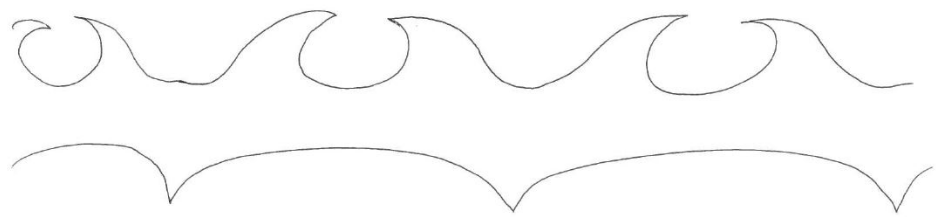
Exercise 2.20 With the above discussion in mind, how would you describe a rotation of an n-dimensional solid in n-space? Hint: compare dimensions of objects.

2.1 Translational Symmetry

The final "classical" symmetry we'll explore is translational symmetry. A plane figure has *translational symmetry* if you can translate the figure by some step-size, and arrive at the same figure. We say "classical" because this, and the two symmetries above, were all considered and studied by mathematicians before the 1900s. The next kind of symmetry (self-similarity) is a relatively modern development.

An example of translational symmetry can be found in the integers $\mathbb{Z}$: translating the integers to the right by 1 unit produces a new copy of $\mathbb{Z}$, where every number n is now at $n+1$. In fact, $\mathbb{Z}$ has many kinds of translational symmetry! Shifting by 2, or 3, or even any integer produces a new copy of $\mathbb{Z}$, just with every point shifted.

A more "artsy" example of translational symmetry is given by the design below, where the pattern extends infinitely to the left and right:



In this case, translating the shape to the left or right, so that one pair of "waves" matches up with a consecutive pair of "waves," is a translational symmetry of the shape.

Exercise 2.21 What kinds of shapes have two different kinds of translational symmetry? I.e. a translational symmetry along one line, together with a translational symmetry along another line.

Exercise 2.22 A triangle, square, or any n-gon does *not* have translational symmetry. Why?

Exercise 2.23 Prove that if a plane figure is bounded, then it *cannot* have translational symmetry.

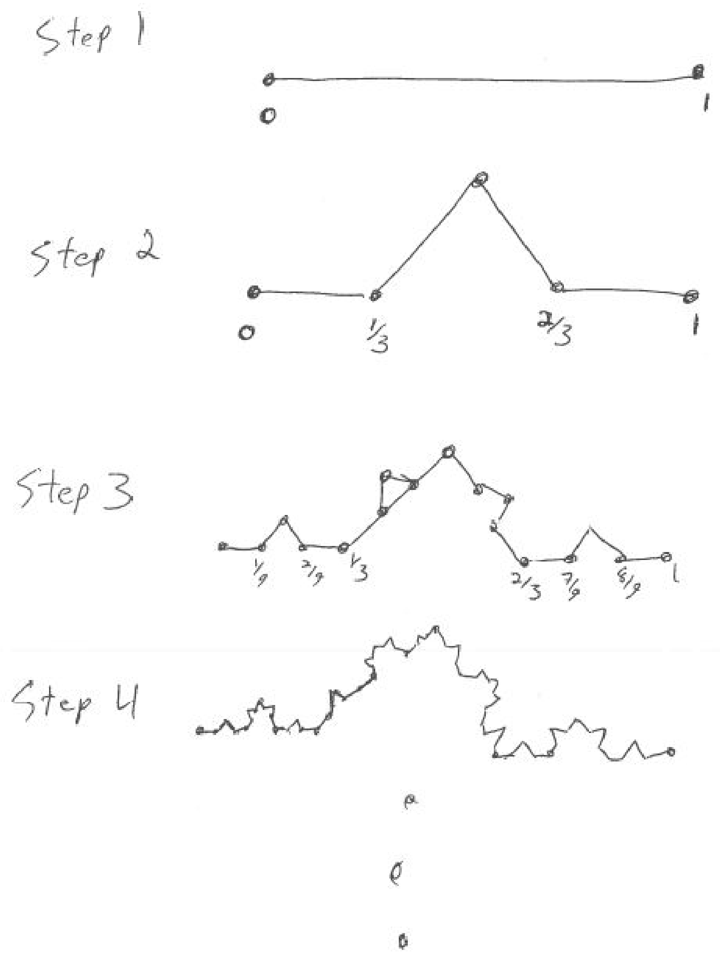
3 Self-symmetry

The next kind of symmetry is self-similarity. We say an object is *self-similar* if it looks the same at different scales. In other words, if we zoom in on the shape then the magnified portion looks exactly the same as the non-magnified version.

The best way to explore self-similarity is by exploring a number of specific examples of self-similar shapes. All of the shapes below are *fractal*s; the name fractal is short for "fractional dimension." In fact most of these shapes have a non-integer dimension, but exploring these details goes beyond this worksheet. For our purposes, we'll be content with exploring these objects and their self-similarity, more so than their dimension.

3.1 Koch Curve

The first fractal is the *Koch curve* (pronounced "Coke curve"). It's constructed inductively, by taking the limit of the following series of shapes:



The triangle in the center is an equilateral triangle.

An interesting fact is that the length of this curve is infinite. To see this, let's compute the length at each step.

In step 1, we just have a unit interval; this has length 1. In step 2, we removed 1/3rd of the interval, but added two intervals each of length 1/3rd. Thus, in step 2 the length is

$$1 - \frac{1}{3} + 2 \cdot \frac{1}{3} = \frac{4}{3}.$$

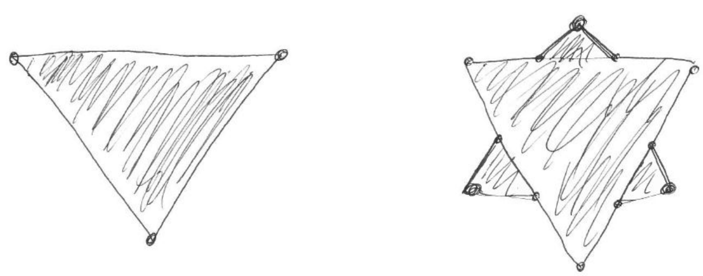
In the third step, we remove four intervals of length 1/9 from our curve of length 4/3, but add eight (which is $2 \cdot 4$) intervals of length 1/9. Thus, the length of the curve at step 3 is

$$\frac{4}{3} - 4 \cdot \frac{1}{9} + 8 \cdot \frac{1}{9} = \frac{16}{9}.$$

Exercise 3.1 What is the length of the curve at the nth step?

Exercise 3.2 Using the last exercise, why is the length of the Koch curve infinite?

In interesting variant of the Koch curve is called the Koch snowflake. This is constructed by taking an equilateral triangle and performing the Koch curve construction on each side of the triangle. The first two steps are drawn below:



Step 1 Step 2.

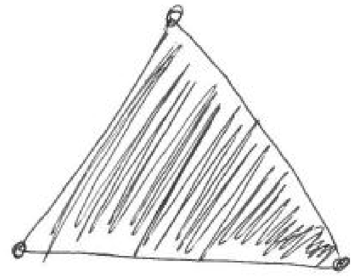
Using the same argument from the previous few exercises, we see that the Koch snowflake has infinite perimeter.

Exercise 3.3 What do you think the area of the Koch snowflake is though? Why?

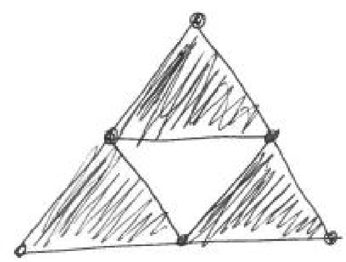
3.2 Sierpinski Triangle

Another self-similar shape we can construct is the *Sierpinski triangle*. To construct this, we take a (filled-in) equilateral triangle with sides of length 1 and remove the middle equilateral triangle. The first few steps are given below:

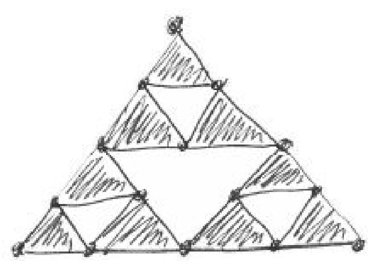
Step 1



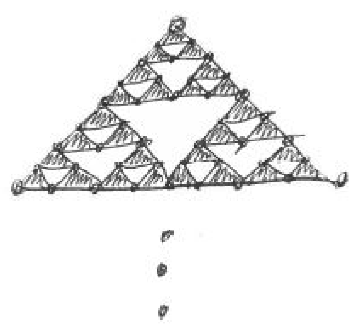
Step 2



Step 3



Step 4



The Sierpinski triangle is what you get if you do this process infinitely many times.

Like the Koch curve, the Sierpinski triangle has infinite perimeter. At the first stage, the perimeter is $1 + 1 + 1 = 3$. At the second stage, we have the outside perimeter, together with the perimeter of the inner triangle. Since the sides of the inner triangle have length $1/2$ (why?), the perimeter of the second step in our construction is

$$3 + 3 \cdot \frac{1}{2} = \frac{9}{2}.$$

At the third step, the smallest triangles have side lengths of $1/4$. Since there are 3 such triangles, and the perimeter of each is $3/4$, the total perimeter is

$$\frac{9}{2} + 3 \cdot \frac{3}{4} = \frac{27}{4}.$$

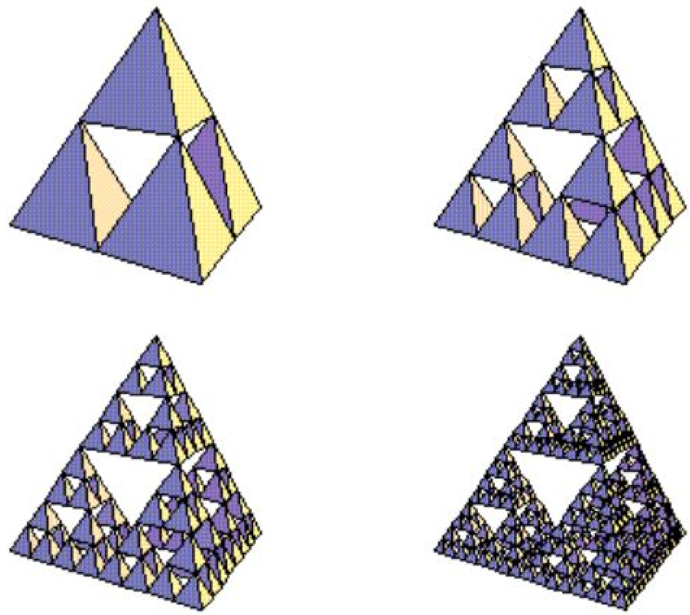
Exercise 3.4 What is the perimeter of this shape at the nth step?

Exercise 3.5 Using the last exercise, why is the perimeter of the Sierpinski triangle infinite?

Exercise 3.6 What is the area of the Sierpinski triangle? Hint: what proportion of the area is removed at each stage of the construction?

Exercise 3.7 We've seen it's possible for a plane figure to have infinite perimeter, but finite area. Is it possible for a plane figure to have infinite area but finite perimeter? Why or why not? (Hint: it is *not* possible.)

A 3-dimensional generalization of the Sierpinski triangle is obtained by taking a tetrahedron, then removing the middle tetrahedron whose vertices are the midpoints of three of the edges of the original tetrahedron, as well as the midpoint of the remaining face. Graphically we have

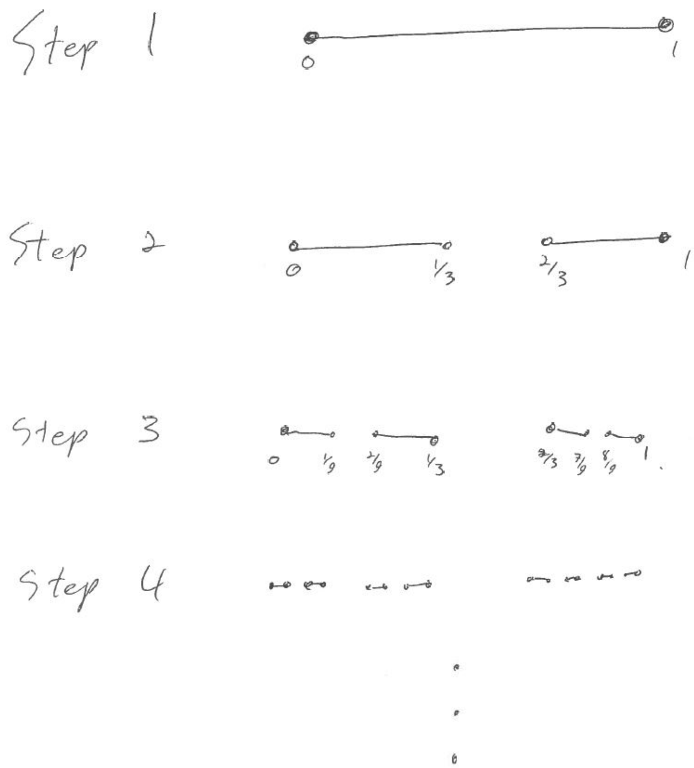


Exercise 3.8 Suppose the starting tetrahedron has side lengths 1. What is the surface area of the 3-d Sierpinski triangle?

Exercise 3.9 *What is the volume of this 3-d fractal?

3.3 Cantor Dust

The last fractal we'll explore is the Cantor set (sometimes called the Cantor Dust). The Cantor set is constructed in the following steps:



In each step of the construction we remove the middle third from each segment in the previous step. The Cantor set is sometimes also called the "middle-thirds set," and this is why.

Exercise 3.10 Prove that the Cantor set has zero length. Hint: what is the area of the set at each stage of the construction?

Exercise 3.11 Despite this, the Cantor set has infinitely many points; why? Hint: at the nth stage, how many endpoints of intervals are there?

Exercise 3.12 *Prove that the Cantor set "has the same number of points" as the interval $[0, 1]$. (Hint: find a connection between points in the Cantor set and binary numbers in $[0, 1]$.)

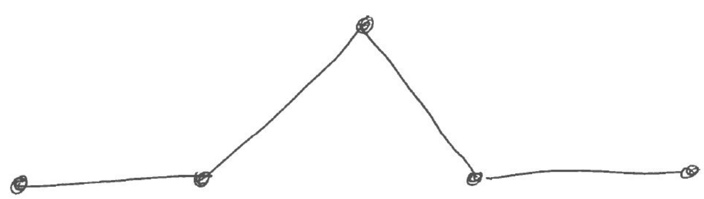
This last exercise illustrates a very counterintuitive result: even though the Cantor set has zero length, it still has infinitely many points. There are other reasons the Cantor set is very interesting to study, but they go far beyond this worksheet.

Exercise 3.13 *We can generalize the Cantor set in the following way: instead of removing the middle-third of each segment, remove the middle-fourth (the segment in the middle with length 1/4 the length of the segment). What is the length of the limiting object in this construction?

3.4 Generating Fractals

In general you can create your own fractal by specifying a generator shape. To construct the fractal one proceeds in steps: the first step is just the generator shape, in the second step you replace every interval/triangle in your generator shape with congruent copies of your generator shape, in the third step you replace intervals/triangles in the second step with congruent copies of the generator shape, etc.

For example, the generator shape of the Koch curve is:



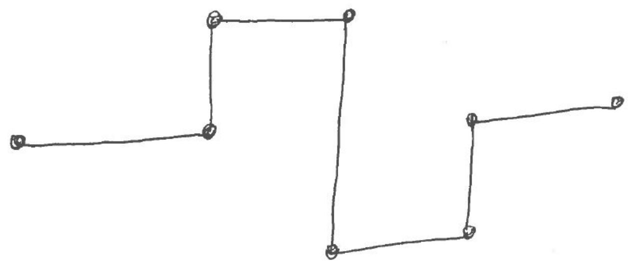
Koch curve generator

The generator shape for the Cantor set is:

Cantor set generator

Exercise 3.14 What is the generator shape for the Sierpinski triangle?

Exercise 3.15 What fractal does this generator shape produce:



Dragon curve generator

Exercise 3.16 Come up with your own generator shapes!