Chapel Hill Math Circle:
Special Kinds of Numbers

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1 Introduction

This worksheet is concerned with three classes of special numbers: the polygonal numbers, the Catalan numbers, and the Lucas numbers. Polygonal numbers are a nice geometric generalization of perfect square integers, Catalan numbers are important numbers that (one way or another) show up in a wide variety of combinatorial and non-combinatorial situations, and Lucas numbers are a slight twist on our familiar Fibonacci numbers.

The purpose of this worksheet is not to give an exhaustive treatment of any of these numbers, since that is just not possible. The goal is to, instead, introduce these interesting classes of numbers and give some motivation to explore on your own.

2 Polygonal Numbers

The first class of numbers we'll consider are called *polygonal numbers*. We'll start with some simpler cases, and then derive a general formula.

Take 3 points and arrange them as an equilateral triangle with side lengths of 2. Here, I use the term "side-length" to mean "how many points are in a side." The total number of points in this triangle (3) is the second triangular number, denoted $P(3, 2)$. To get the third triangular number $P(3, 3)$, we take the last configuration and adjoin points so that the outermost shape is an equilateral triangle with a side-length of 3. To do this, we can add 3 points along any one of the edges. Since there are 3 points in the original

triangle, and we added 3 points to get the next triangle, the total number of points in this step is $P(3,3) = 3 + 3 = 6$.

Exercise 2.1 What is $P(3,4)$? What about $P(3,5)$? What about $P(3,1)$?

Exercise 2.2 In general, what is $P(3,n)$? Is this number related to another familiar formula you recognize?

Next we consider the family of numbers $P(4,n)$, called the *square numbers*.

Exercise 2.3 Using the same ideas as for $P(3,n)$, what do you think 4 and n represent in the expression $P(4,n)$?

To compute $P(4,2)$, we need to take a number of points and arrange them into a square of side length 2. In other words, we are computing the area of a square with side lengths 2; this gives $P(4,2) = 2^2 = 4$.

Exercise 2.4 What picture would you draw (similar to the construction for $P(3,3)$) to represent $P(4,2)$? Draw a similar picture for $P(4,3)$ and verify that $P(4,3) = 9$.

Exercise 2.5 In general, what is $P(4,n)$?

The next family consists of *pentagonal numbers*, and we denote the nth pentagonal number by $P(5,n)$. As the name suggests, these numbers count the number of points required to form a regular pentagon with side lengths n.

To construct a pentagon with side length 2, we will need a total of $P(5,2) = 5$ points. The next pentagon is constructed by taking this last shape and adding enough points so that the outermost pentagon is regular, with side lengths 3.

By looking at the picture, we can count the total number of points to be $P(5,3) = 12$. We can also realize this number as

$$P(5,3) = P(5,2) + 7,$$

since 7 points need to be added to form the remaining sides of the new pentagon.

Exercise 2.6 How many points do you need to form the pentagon for $P(5,4)$ from the pentagon for $P(5,3)$? Using this, what is $P(5,4)$?

Exercise 2.7 How many points do you need to go from the shape corresponding to $P(5,4)$ to the shape for $P(5,5)$? In general, how many points do you need to add to get from $P(5,n)$ to $P(5,n+1)$?

Exercise 2.8 What is $P(5,n)$, where n is any integer?

Finally, we'll work with the *hexagonal numbers* $P(6,n)$.

Exercise 2.9 What shape corresponds to $P(6,2)$? What about $P(6,3)$? Use these to compute $P(6,2)$ and $P(6,3)$.

Exercise 2.10 How many points do you need to add the shape for $P(6,2)$ to get the shape for $P(6,3)$? What about between $P(6,3)$ to $P(6,4)$?

Exercise 2.11 If you have the shape for $P(6,n)$, how many points do you need to add to get the shape for $P(6,n+1)$?

Exercise 2.12 What is $P(6,n)$, for any n?

With all of these special cases under our belt, we can now tackle the general case. The *n*th *s-polygonal number*, denoted $P(s,n)$, is the number of points required to draw a regular s-polygon with side length n.

Exercise 2.13 $P(s,1) = 1$ and $P(s,2) = s$ for any s; why?

Exercise 2.14 What is the difference $P(s,n+1) - P(s,n)$ as a function of s and n? What does this difference represent geometrically?

Exercise 2.15 Using the last exercise (or otherwise), what is $P(s,n)$?

Exercise 2.16 Show that $P(s,n) = (s-3)P(3,n-1) + P(3,n)$.

Exercise 2.17 Using the last exercise, show that $P(s+1,n) - P(s,n) = P(3,n)$.

Exercise 2.18 It turns out that every hexagonal number is also a triangular number. Use the formula for $P(s,n)$ in terms of triangular numbers to show that $P(6,n) = P(3,2n-1)$.

3   Catalan Numbers

Catalan numbers are important in combinatorics, the field of math concerned with counting, as well as many other seemingly unrelated areas of math. In this section we'll introduce this family of numbers and derive some basic properties of them.

Suppose you have 2 left- and right-curved brackets: ((and)). The ways we can arrange these brackets so that any left-bracket (is matched with a right-bracket) are the following:

$$(()), ()().$$

Thus there are only 2 ways to arrange these 4 brackets, and the second Catalan number $C_2 = 2$.

Now suppose you have 3 left- and right-curved brackets: (((and))). Let C_3 denote the number of ways you can arrange these brackets so that a left-bracket always has a corresponding right-bracket to its right, as above. Two viable arrangements are ((())) and ()()(). It turns out there are 3 more viable arrangements, so $C_3 = 5$.

Exercise 3.1 What are the 3 other arrangements?

It's not too hard to see that $C_1 = 1$. For our purposes it's convenient to let $C_0 = 1$ as well; you can think of this as a definition more so than a property of the numbers, but it's similar to the statement "every human on Mars has three eyes." Since there are no humans on Mars, it is automatically true that each one of them has three eyes. Pursuing this discussion would take us closer to logic than combinatorics though, so let's get back to Catalan numbers.

Exercise 3.2 How many ways can you arrange 4 left- and 4 right-curved parenthesis? Use this to compute C_4.

After C_4, the Catalan numbers start to grow a bit faster:

$$C_5 = 42,$$
$$C_6 = 132,$$
$$C_7 = 429,$$
$$C_8 = 1430,$$
$$C_9 = 4862,$$

etc. There turn out to be a few closed form representations for C_n which make computing these numbers slightly easier, such as

$$C_n = \frac{(2n)!}{(n+1)!n!},$$

where $n! = n \cdot (n-1) \cdots \cdots 3 \cdot 2 \cdot 1$. We won't prove this, but we will derive a recurrence relation for the Catalan numbers, i.e. a method to compute C_{n+1} from $C_n, C_{n-1}, ..., C_2, C_1$, and C_0.

Exercise 3.3 How can you get each of the configurations used to compute C_3 from one of the configurations for C_2 and C_1?

Exercise 3.4 How can you get each of the configurations used to compute C_4 from the configurations for C_3, C_2, and C_1?

Exercise 3.5 Using ideas from the last exercise (or otherwise), show that Catalan numbers satisfy the recursion relation

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{i-1}.$$

Exercise 3.6 Using the recurrence relation, verify that $C_5 = 42$ and $C_6 = 132$.

The next few exercises are meant to give an indication of how prevalent Catalan numbers are.

Exercise 3.7 Show that C_n counts the number of ways to multiply (in order) $n + 1$ terms. For example, if we have 3 factors x, y, z, then the possibilities are

$$(x \cdot y) \cdot z \text{ or } x \cdot (y \cdot z),$$

and $C_2 = 2$. Hint: reinterpret multiplication as distributing parentheses.

Exercise 3.8 Show that C_n counts the number of ways a regular polygon with $n+2$ sides can be cut into triangles, with the cuts made between vertices, and so that no cuts intersect. For example, we can cut a square (4 sides) in $2 = C_2$ of these ways by cutting along either of the diagonals. Hint: verify this with a regular pentagon, i.e. show a pentagon can be cut into triangles in C_3 ways by producing every possible arrangement.

4 Fibonacci and Lucas Numbers

Recall that the *Fibonacci numbers* F_n are constructed by setting $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$.

Exercise 4.1 As a refresher, write down the first 6 Fibonacci numbers, i.e. write down $F_1, F_2, ..., F_5, F_6$.

In this section, we'll explore a slight variant of the Fibonacci numbers called the *Lucas numbers* L_n. The construction goes as follows: let $L_1 = 1$, $L_2 = 3$, and in general $L_n = L_{n-1} + L_{n-2}$.

Exercise 4.2 Write down the first 6 Lucas numbers. Are they the same as the Fibonacci numbers?

Suppose we tried to extend this sequence in the other direction, for $n \leq 0$. The recurrence relation tells us that

$$L_2 = L_1 + L_0$$
$$3 = 1 + L_0$$
$$2 = L_0,$$

and similarly

$$L_1 = L_0 + L_{-1}$$
$$1 = 2 + L_{-1}$$
$$-1 = L_{-1},$$

so $L_0 = 2$, and $L_{-1} = -1$.

Exercise 4.3 Using the method above, compute $L_{-2}, L_{-3}, ..., L_{-6}$. What pattern do you notice?

Exercise 4.4 What happens if you try the same construction with the Fibonacci numbers? I.e. what are F_0, F_{-1}, etc.?

Since Lucas and Fibonacci numbers are constructed in a similar way, it is reasonable to suspect there are some identities relating the two numbers. This is indeed the case!

Exercise 4.5 Verify the identities

$$L_n = F_{n-1} + F_{n+1} = F_n + 2F_{n-1} = F_{n+2} - F_{n-2}.$$

Hint: write the first few Lucas numbers in terms of Fibonacci numbers, and find a pattern.