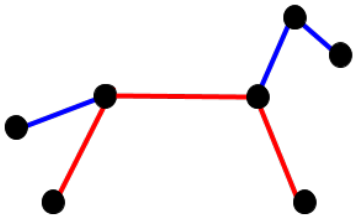
Hackenbush

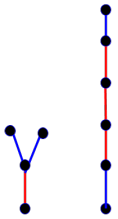
1 Warm-Up

In Red-Blue Hackenbush, there are two players named Red and Blue who alternate moves. Red moves by cutting a red segment and Blue, by cutting a blue one. The cut segment is deleted together with any other segments that are no longer connected to the ground.
A player loses when they can't make a move (because there are no longer any segments of their color).
For example, in the (very boring) game below, made up of two "bushes", Red can cut from either bush, and Red will always win, no matter what moves are made, and no matter who goes first.

1. Try playing the game below. What are the best moves? Is this game position better for Red or Blue?

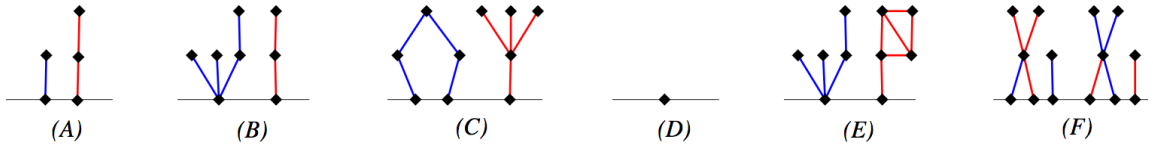


2. What about this game?

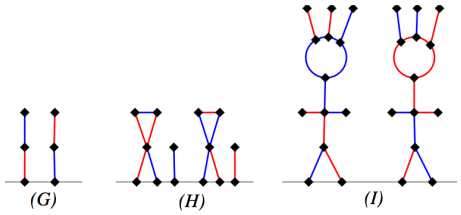


2 Games

3. For each of these games, decide who wins if Blue goes first and who wins if Red goes first, assuming optimal play. Which game is the "most winning" for Red? Why?



4. Analyze who wins in these games. How does the symmetry help?



3 Numbers

5. To evaluate which games are winning for Red and which for Blue, and to decide on the best moves, it will be helpful to assign a number to each Hackenbush game. We will look for numbers that satisfy the following conditions:

   (a) Positions that are better for Blue will get positive numbers.

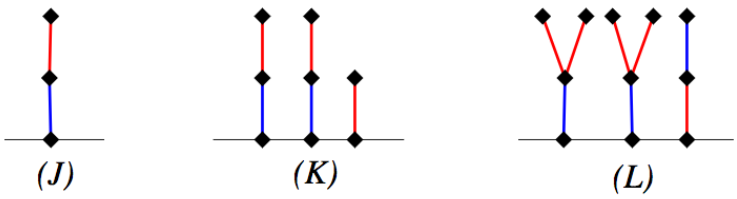
   (b) Positions that are better for Red will get negative numbers.

   (c) Positions that are equally good for Blue and Red will get the number 0.

   (d) If we put two positions together, their numbers should add.

6. What numbers would you give to positions (A) - (I) above?

7. Is position (J) positive or negative or zero? What number should it get?

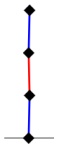


8. Decide if positions (K) and (L) are better for Blue or Red, and use that to assign a number to each of the stalks of (K) and (L).

4 Surreal Numbers

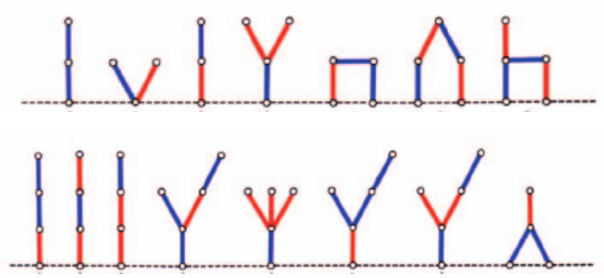
One way to assign a number to a hackenbush position is to look at the positions that could result after the first move.

9. In this figure, if we erase a blue segment in all possible ways, what are the numbers of the positions that result? What about if we erase a red segment in all possible ways?

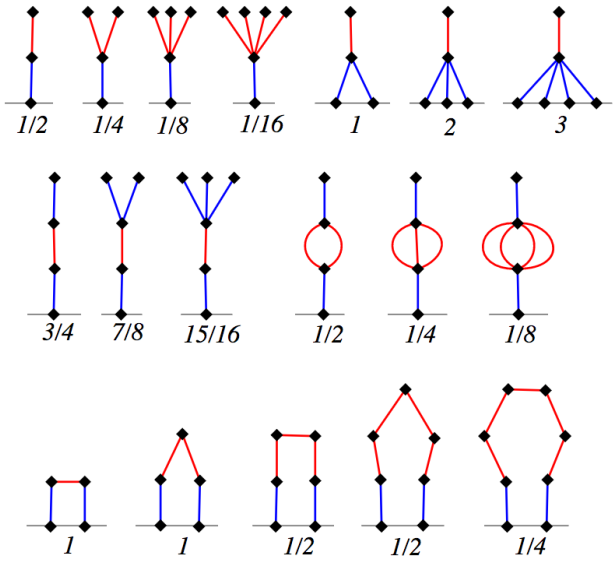


10. What number should we use for this position?

11. Use this method, to assign numbers to the positions in this figure:



12. Below are some hackenbush numbers. Use some of these pieces and their "negatives", or others, to put together a game that is winnable for Blue but hard to win. Try playing it with your neighbor.



5 Infinite Hackenbush

13. Suppose there are two infinitely-long stalks, one all of red segments and the other all blue.
    (a) Is this game better for Red, better for Blue, or equal?
    (b) What is the best strategy?
14. Suppose there are two infinitely-long stalks, one all of red segments and the other all blue, together with one single red segment. If Red goes first, what is Red's best move?
15. What number should be assigned to an infinitely-long stalk that is entirely red? What number should be assigned to an infinitely long stalk that is entirely red, together with one single red segment?
16. What number should be assigned to an infinitely long stalk with one blue segment at the bottom and the rest red? Hint: consider finite stalks that contain a single blue segment at the bottom, and a longer and longer sequence of red segments above it, How many of these stalks would be needed to perfectly balance out a single red segment to make a 0 game?
17. What number should be assigned to an infinitely-long stalk of alternating blue and red segments, starting wtih blue? Build a balanced 0 game with some of these stalks and some single segments, and try playing it with your neighbor. .

6 Red Blue Green Hackenbush

In Red Blue Green Hackenbush, green segments can be taken by either player. Try playing the one flower game, the two flower game, and the flower garden and house games. (p. 31). What numbers would you assign to these games? How is Red Blue Green Hackenbush different from Red Blue Hackenbush?