Recurrence Relations

Definitions and Introductory Problems

A **recurrence relation** is an equation that defines a few initial values and then recursively defines a sequence where each further term is given as a function of the preceding terms.

A **recursive definition** is when a thing is defined in terms of itself or its own type. (Hence the joke "In order to understand recursion, one must first understand recursion")

For example, the following is a recurrence definition for a sequence:

$$\begin{cases} a_1 &= 1 \\ a_n &= a_{n-1} + 2 \end{cases}$$

Can you write out the first few terms of the sequences?

n	1	2	3	4	5	6	7	8	9	10
a_n	1									

A **closed form** expression of a recurrence is an equivalent definition for the sequence that does not involve referring back to itself. This is usually done as a function of the index in the sequence. For example, the closed form of the above sequence is:

$$a_n = 1 + 2(n-1)$$

Practice Problems

1. Find the next 3 elements in the sequence, and then write out the recurrence relationship.
   (a) 2, 4, 8, 16, 32,
   (b) 3, 7, 15, 31, 63,
2. For 1.a., can you find a closed form representation for each of the sequences?

But sometimes the recurrence depends on more than just the previous term, consider the famous **Fibonacci Sequence**, given by the recurrence:

$$\begin{cases} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \end{cases}$$

3. Write out the first few terms of the Fibonacci sequence.

4. Consider the following sequence:

$$2, 1, 3, 4, 7, 11, 18, 29, ...$$

What is its recurrence relationship?

But recurrence appear beyond number sequences, they are everywhere around us, consider the following "real life" problems:

5. If a "word" is defined as any collection of letters, how many n-letter words are there that only have letters "A" and "B". Such words include "A", "ABA", "AABBAB", etc.

   (a) To start, how many single letter words are there of this type?

   (b) What about 2 letters? 3 letters? Is there a pattern

   (c) Consider a n-letter word, how many choices do you have for the last letter? How does the total count relate to the total amount of $(n-1)$-letter word?

6. How many ways are there to fill up a $2 \times n$ board with 1×2 dominoes?

   (a) To start let us consider how many ways are there to fill up a 2×1 board with dominoes.

(b) What about a 2×2 board? 2×3?

(c) What about $2 \times n$? (Think about it as how can we use previously filled boards to fill up a $2 \times n$ board)

7. (Challenge Problem) Find a closed form representation for 1.b.

8. (Challenge Problem) Last time we meet we discussed the concept of combinations, or ${}_nC_k$, prove, (at least intuitively), that ${}_nC_k =_{n-1} C_{k-1} +_{n-1} C_k$.