Polyhedra

1 Warm-up Problems

1. The table below is a partial list of the number of vertices, edges, and faces (V, E, F) of some polyhedra. Fill in the missing entries.

V	E	F
8		6
6		8
4		6
6		4
5	8	
6		6
7	15	
12	30	
24	36	

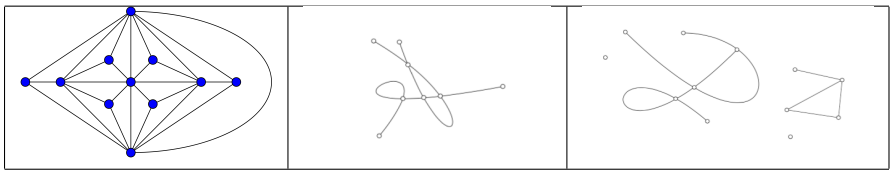
2. For each row of the table, can you find some polyhedron that has the corresponding numbers V, E, F? For example, the first row might be a cube.

3. Suppose a polyhedron has 60 faces, all of them triangles. What are V, E, and F?

2 Proof of Euler's Formula

See the Geometry Junkyard by David Epstein https://www.ics.uci.edu/ eppstein/junkyard/euler/ for 20 proofs of Euler's Formula. One of my favorites, which is not on this list, proves the formula by first making it *more general*.

4. A planar graph is a collection of vertices and edges drawn in the plane, in such a way that there is a vertex at the beginning and end of each edge and at any place where two edges meet or cross. Edges do NOT have to be straight. Here are a few examples.



What patterns hold for the number of faces, edges, and vertices?

5. Prove that the patterns you found hold for all planar graphs with 0 edges. Prove it for planar graphs with 1 edge. Prove it for planar graphs with n edges, for any n.

3 Census of Polyhedra

The following table gives the number of distinct polyhedra for each number of vertices and faces.

Number of Polyhedra														
Total (row)	*Vertex Count* in row and *Face Count* in column													
96262938	**13**									219	7916	104213	709302	2937495
6384634	**12**								14	558	8822	64439	268394	709302
440564	**11**								38	768	6134	25626	64439	104213
32300	**10**							5	76	633	2635	6134	8822	7916
2606	**9**							8	74	296	633	768	558	219
257	**8**						2	11	42	74	76	38	14	
34	**7**						2	8	11	8	5			
7	**6**					1	2	2	2					
2	**5**					1	1							
1	**4**				1									
0	**3**													
0	**2**													
0	**1**													
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**

From http://www.numericana.com/data/polyhedra.htm.

6. Why is the table symmetric across the line $y = x$?

7. Why do all the non-zero entries lie in a wedge-shaped region? What are the slopes of the lines that bound the wedge? Explain.

4 Problems involving Euler Characteristic from last time

8. In a certain small country there are villages, expressways, and fields. Expressways only lead from one village to another and do not cross one another, and it is possible to travel from any village to any other village along the expressways. Each field is completely enclosed by expressways and villages. If there are ten villages and sixteen expressways, then how many fields are there in this country?

9. A polyhedron is made up of pentagons and hexagons . How many pentagons must there be? Prove that no other number of pentagons is possible.

10. A polyhedron is built entirely from triangles, in such a way that 5 faces meet at each vertex. Prove that it has to have 20 faces. (Hint: first deduce that 3F = 2E and 3F = 5V)

11. A *Platonic solid* is a polyhedron whose faces are all the same regular polygon (for example, all equilateral triangles or all squares) and for which each vertex has the same number of faces meeting at it (for example, 3 faces meet at each vertex). For example, a cube is a Platonic solid because all its faces are squares and every vertex has exactly three squares around it. But if we glued two cubes together and looked at the polyhedron made from the exposed faces, it would not be a Platonic solid because some vertices would have 3 squares around them and some would have 4.

    How many platonic solids are there? Prove it! Hint: use the previous problem, and other cases like it.

12. Can you solve the utilities problem or prove that it cannot be done?

13. Can you solve the problem of drawing 5 vertices and an edge between each pair of vertices, in such a way that no edges cross?

Thanks to Matt Kahle for many of these problems. Thanks also to Sam Vandervelde's *Circle in a Box* and Tom Davis' website *www.geometer.org* for some of the problems.